A SUBSET SEMANTICS FOR THE WEAK GÖDEL MODAL LOGICS

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Abstract. We construct, inspired by the recent *subset semantics* for classical justification logics by Lehmann and Studer, a novel possible world semantics for the weak Gödel modal logics (weakenings of the standard Gödel modal logics of Caicedo and Rodriguez). In this semantics, the interpretation of the $\Box$-modality *does not* depend on the syntactic structure of the following formula, a property which all the previous semantics for the weak Gödel modal logics do not have. Further, this semantics at the same time naturally generalizes the semantics of Caicedo and Rodriguez for the standard Gödel modal logics based on Gödel-Kripke models.

1. Introduction and preliminaries

In [9], Lehmann and Studer presented a new semantics for classical justification logics called *subset semantics*. They give a novel possible-worlds model construction where the interpretation of formulas $t : \phi$ *does not* (directly) depend on the syntactical structure of ϕ but just on that of t.[1] This deviates from the usual style of semantics for justification logics (see e.g. [4, 5, 10]) where there is commonly a function directly interpreting formulas of the kind $t : \phi$ by hard coding truth-values (based on the syntactic structure of *both* ϕ *and* t).

In this note, we consider not justification logics but non-explicit modal logics based on the usual modality $\Box$, and this in a many-valued setting with the so called *weak Gödel modal logics*, introduced in [12]. These many-valued modal logics arise by omitting a specific axiom scheme from the standard Gödel modal logics from [2]. The motivation for those standard Gödel modal logic is of semantical nature, axiomatizing the semantical consequence relation based on the class of so-called *Gödel-Kripke* models which are natural generalizations of the classical Kripke models to values in the interval $[0, 1]$, in relation to the $[0, 1]$-valued propositional Gödel logic, originally defined in [3] (building on earlier work of Gödel in [7], see also [1, 8, 14]).

An immediate question is of how these weak Gödel modal logics can be semantically captured as the intended semantics for the standard Gödel modal logics from [2], based on the aforementioned Gödel-Kripke models, appears in full generality. However, following their origin in [12], the weak Gödel modal logics turn out to have a close connection to the Gödel justification logics from [6, 11], and (utilizing this close connection) a completeness theorem for the weak Gödel modal logics with respect to a semantics *transferred* from the Gödel justification logics was established in [12]. In this semantics however, the interpretation of formulas of the form $\Box\phi$ *does* directly depend on the syntactic structure of ϕ.

The above mentioned subset models, giving rise to a semantics for the classical justification logics, can be generalized to this many-valued setting of Gödel justification logics (as established in the forthcoming [13]). We follow the line of transferring semantics from Gödel justification logics to the weak Gödel modal logics and, in this note, give a novel many-valued possible-worlds semantics for the weak Gödel modal logics which is transferred from the above mentioned (many-valued generalization of the) classical subset models. In this transferred semantics, the truth value of a formula of the form $\Box\phi$ *does not* (directly) depend on the syntactic structure of ϕ. These results, although inspired from results *for* (Gödel) justification logics, are stated here without any further reference to any concepts from this context.

Throughout the note, we consider the language

$$\mathcal{L}_\Box : \phi ::= \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid \Box\phi$$

for $p \in Var := \{p_i \mid i \in \mathbb{N}\}$. The formula $\neg\phi$ is considered to be an abbreviation for $(\phi \to \bot)$.

Definition 1. Over $\mathcal{L}_\Box$, we define the following proof calculi:

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[1]The semantics does rely on introducing "impossible" worlds to retain hyperintensionality. These impossible worlds do not have to interpret formulas truth-functionally and thus can be seen as being more syntactically flavoured.

(1) $\mathcal{GK}_\Box^-$ as the extension of a set of axiom schemes for propositional Gödel logic, $\mathcal{G}$ (see e.g. [1, 3, 14]), with the axiom scheme

$$(K) : \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

together with the rules modus ponens $(MP) : \phi, \phi \to \psi/\psi$ and necessitation $(R\Box) : \vdash \phi/ \vdash \Box\phi$;

(2) $\mathcal{GT}_\Box^-$ is $\mathcal{GK}_\Box^-$ extended by the axiom scheme $(T) : \Box\phi \to \phi$;

(3) $\mathcal{GK}4_\Box^-$ is $\mathcal{GK}_\Box^-$ extended by the axiom scheme $(4) : \Box\phi \to \Box\Box\phi$;

(4) $\mathcal{GS}4_\Box^-$ is $\mathcal{GK}_\Box^-$ extended by the schemes (T) and (4).

In the following, let $\mathcal{GML}_\Box^- \in \{\mathcal{GK}_\Box^-, \mathcal{GT}_\Box^-, \mathcal{GK}4_\Box^-, \mathcal{GS}4_\Box^-\}$. We write $\Gamma \vdash_{\mathcal{GML}_\Box^-} \phi$ for derivability of ϕ in this calculus under the assumptions Γ which is defined as usual with Hilbert-type systems. We also write $\vdash_{\mathcal{GML}_\Box^-} \phi$ for $\emptyset \vdash_{\mathcal{GML}_\Box^-} \phi$. Note, that the notation of the rule $(R\Box)$ is chosen as to indicate that it may only be applied to *pure theorems* of the system in a derivation.

A first observation is that all of these logics enjoy the classical deduction theorem.

Lemma 2 (Deduction theorem)**.** *For any* $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_\Box$*:* $\Gamma \cup \{\phi\} \vdash_{\mathcal{GML}_\Box^-} \psi$ *iff* $\Gamma \vdash_{\mathcal{GML}_\Box^-} \phi \to \psi$.

The proof is a straightforward generalization of the classical case.

2. Many-Valued Subset Models

The semantics we introduce is inspired by the subset models for classical justification logics by Lehmann and Studer. We however consider (based on the choice of Gödel logic as the base logic) many-valued generalizations taking truth-values in the interval $[0, 1]$, as said before. For this, we consider the following binary operations $\odot, \oplus, \Rightarrow$ on $[0, 1]$:

- $\odot : (x, y) \mapsto \min\{x, y\}$;
- $\oplus : (x, y) \mapsto \max\{x, y\}$;
- $\Rightarrow : (x, y) \mapsto \begin{cases} 1 & \text{if } x \leq y; \\ y & \text{otherwise.} \end{cases}$

We also consider the derived function $\sim x := x \Rightarrow 0$ $(x \in [0, 1])$ and write $\sim^2 x$ for $\sim\sim x$. Using the definition of $\Rightarrow$, one obtains the following for $\sim$ and $\sim^2$:

$$\sim x = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad \sim^2 x = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3. By a *Gödel-Subset model*, we mean a structure $\mathfrak{M} = \langle \mathcal{W}, \mathcal{W}_0, \mathcal{R}, \mathcal{V} \rangle$ with

(1) $\mathcal{W} \neq \emptyset$,
(2) $\mathcal{W}_0 \subseteq \mathcal{W}$, $\mathcal{W}_0 \neq \emptyset$,
(3) $\mathcal{R} : \mathcal{W} \times \mathcal{W} \to [0, 1]$,
(4) $\mathcal{V} : \mathcal{W} \times \mathcal{L}_\Box \to [0, 1]$,

such that $\mathcal{V}(w, \cdot)$ satisfies

(i) $\mathcal{V}(w, \bot) = 0$,
(ii) $\mathcal{V}(w, \phi \land \psi) = \mathcal{V}(w, \phi) \odot \mathcal{V}(w, \psi)$,
(iii) $\mathcal{V}(w, \phi \lor \psi) = \mathcal{V}(w, \phi) \oplus \mathcal{V}(w, \psi)$,
(iv) $\mathcal{V}(w, \phi \to \psi) = \mathcal{V}(w, \phi) \Rightarrow \mathcal{V}(w, \psi)$,
(v) $\mathcal{V}(w, \Box\phi) = \inf\{\mathcal{R}(w, v) \Rightarrow \mathcal{V}(v, \phi) \mid v \in \mathcal{W}\}$.

for all $w \in \mathcal{W}_0$ and all $\phi, \psi \in \mathcal{L}_\Box$ and such that it is *regular*, that is it satisfies the following:

(a) $\mathcal{R}(w, v) \leq \inf_{\psi \in \mathcal{L}_\Box}\{\sup_{\phi \in \mathcal{L}_\Box}\{\mathcal{V}(v, \phi \to \psi) \odot \mathcal{V}(v, \phi)\} \Rightarrow \mathcal{V}(v, \psi)\}$ for all $w \in \mathcal{W}_0$ and all $v \in \mathcal{W}$;
(b) for every $\phi \in \mathcal{L}_\Box$, if $\mathcal{V}(w, \phi) = 1$ for all $w \in \mathcal{W}_0$, then $\mathcal{V}(w, \phi) = 1$ for all $w \in \mathcal{W}$.

We denote the class of all Gödel-Subset models by GS. Given a model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{W}_0, \mathcal{R}, \mathcal{V} \rangle$, we also write $\mathcal{D}(\mathfrak{M}) := \mathcal{W}$ and $\mathcal{D}_0(\mathfrak{M}) := \mathcal{W}_0$. We write $(\mathfrak{M}, w) \models \phi$ if $\mathcal{V}(w, \phi) = 1$ and $(\mathfrak{M}, w) \models \Gamma$ if $(\mathfrak{M}, w) \models \gamma$ for all $\gamma \in \Gamma$ where $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$.

Remark 4. By the definition of $\Rightarrow$, it holds that

$$x \leq y \Rightarrow z \text{ iff } x \Rightarrow y \leq x \Rightarrow z$$

for all $x, y, z \in [0,1]$. Therefore, we have

$$\begin{aligned}
\mathcal{R}(w,v) &\leq \inf_{\psi\in\mathcal{L}_\Box}\{\sup_{\phi\in\mathcal{L}_\Box}\{\mathcal{V}(v,\phi\to\psi)\odot\mathcal{V}(v,\phi)\}\Rightarrow\mathcal{V}(v,\psi)\} \\
&\text{iff } \mathcal{R}(w,v) \leq \sup_{\phi\in\mathcal{L}_\Box}\{\mathcal{V}(v,\phi\to\psi)\odot\mathcal{V}(v,\phi)\}\Rightarrow\mathcal{V}(v,\psi) \text{ for all } \psi\in\mathcal{L}_\Box \\
&\text{iff } \mathcal{R}(w,v)\Rightarrow\sup_{\phi\in\mathcal{L}_\Box}\{\mathcal{V}(v,\phi\to\psi)\odot\mathcal{V}(v,\phi)\}\leq\mathcal{R}(w,v)\Rightarrow\mathcal{V}(v,\psi) \text{ for all } \psi\in\mathcal{L}_\Box.
\end{aligned}$$

We may introduce different, more refined model classes than GS by imposing various restrictions on R, similar to the common restrictions on Gödel-Kripke models in the context of the standard Gödel modal logics.

Definition 5. Let $\mathfrak{M} = \langle\mathcal{W},\mathcal{W}_0,\mathcal{R},\mathcal{V}\rangle$ be a GS-model. $\mathfrak{M}$ is called:

(1) *reflexive* if for all $w\in\mathcal{W}_0$: $\mathcal{R}(w,w) = 1$;
(2) *transitive* if for all $w\in\mathcal{W}_0, v\in\mathcal{W}$: $\mathcal{R}(w,v)\leq\inf\{\mathcal{V}(w,\Box\phi)\Rightarrow\mathcal{V}(v,\Box\phi)\mid\phi\in\mathcal{L}_\Box\}$;
(3) *accessibility-crisp* if for all $w,v\in\mathcal{W}:\mathcal{R}(w,v)\in\{0,1\}$.

We denote the class of all reflexive, transitive or reflexive and transitive models by GST, $\mathsf{GSK4}$ or $\mathsf{GSS4}$, respectively. Given a class C of GS-models, we denote the subclass of all accessibility-crisp models in C by C^{c}.

As common in the context of Gödel (modal) logics, there are now two natural choices for semantics consequence.

Definition 6. Given $\Gamma\cup\{\phi\}\subseteq\mathcal{L}_\Box$ and a class C of GS-models, we define:

(1) $\Gamma\models_{\mathsf{C}}\phi$ iff $\forall\mathfrak{M} = \langle\mathcal{W},\mathcal{W}_0,\mathcal{R},\mathcal{V}\rangle\in\mathsf{C}\forall w\in\mathcal{W}_0\,(\mathcal{V}(w,\Gamma)\leq\mathcal{V}(w,\phi))$;
(2) $\Gamma\models^1_{\mathsf{C}}\phi$ iff $\forall\mathfrak{M} = \langle\mathcal{W},\mathcal{W}_0,\mathcal{R},\mathcal{V}\rangle\in\mathsf{C}\forall w\in\mathcal{W}_0\,((\mathfrak{M},w)\models\Gamma$ implies $(\mathfrak{M},w)\models\phi)$.

Here, we write $\mathcal{V}(w,\Gamma) := \inf\{\mathcal{V}(w,\gamma)\mid\gamma\in\Gamma\}$.

Naturally, (1) implies (2). It will turn out that these different forms of semantic consequence are (at least for some model classes) equivalent. This is a phenomenon which originates already on the purely propositional level (see e.g. [1, 14]) and is also present in the context of the standard Gödel modal logics (see [2]).

We obtain a soundness result for any choice of $\mathcal{GML}^-_\Box$ and the corresponding model classes. For this, let GSML be the class of GS-models corresponding to $\mathcal{GML}^-_\Box$.

Lemma 7. *For all $\Gamma\cup\{\phi\}\subseteq\mathcal{L}_\Box$, we have $\Gamma\vdash_{\mathcal{GML}^-_\Box}\phi$ implies $\Gamma\models_{\mathsf{GSML}}\phi$.*

Proof. Once we have verified that $\vdash_{\mathcal{GML}^-_\Box}\phi$ implies $\models_{\mathsf{GSML}}\phi$, we immediately obtain the strong claim as

$$\begin{aligned}
\Gamma\vdash_{\mathcal{GML}^-_\Box}\phi \text{ impl. } &\{\gamma_1,\dots,\gamma_n\}\vdash_{\mathcal{GML}^-_\Box}\phi \\
\text{impl. } &\vdash_{\mathcal{GML}^-_\Box}\bigwedge_{k=1}^n\gamma_k\to\phi \\
\text{impl. } &\models_{\mathsf{GSML}}\bigwedge_{k=1}^n\gamma_k\to\phi \\
\text{impl. } &\forall\mathfrak{M} = \langle\mathcal{W},\mathcal{W}_0,\mathcal{R},\mathcal{V}\rangle\in\mathsf{GSML}\forall w\in\mathcal{W}_0 \\
&\quad(\mathcal{V}(w,\Gamma)\leq\min\{\mathcal{V}(w,\gamma_k)\mid 1\leq k\leq n\}\leq\mathcal{V}(w,\phi))
\end{aligned}$$

using the deduction theorem where the last line is $\Gamma\models_{\mathsf{GSML}}\phi$. To verify the weak soundness claim, we only check that the modal axioms are valid in their respective model classes. For this, let $\mathfrak{M} = \langle\mathcal{W},\mathcal{W}_0,\mathcal{R},\mathcal{V}\rangle$ be a GSML-model and $w\in\mathcal{W}_0$.

For the axiom scheme (K), let $\phi,\psi\in\mathcal{L}_\Box$. We have

$$\begin{aligned}
\mathcal{V}(w,\Box(\phi\to\psi))\odot\mathcal{V}(w,\Box\phi) &\leq(\mathcal{R}(w,v)\Rightarrow\mathcal{V}(v,\phi\to\psi))\odot(\mathcal{R}(w,v)\Rightarrow\mathcal{V}(v,\phi)) \\
&\leq\mathcal{R}(w,v)\Rightarrow(\mathcal{V}(v,\phi\to\psi)\odot\mathcal{V}(v,\phi)) \\
&\leq\mathcal{R}(w,v)\Rightarrow\sup_{\phi\in\mathcal{L}_\Box}\{\mathcal{V}(v,\phi\to\psi)\odot\mathcal{V}(v,\phi)\} \\
&\leq\mathcal{R}(w,v)\Rightarrow\mathcal{V}(v,\psi)
\end{aligned}$$

for any $v\in\mathcal{W}$ where the last line follows from Remark 4. Thus, by taking the meet over v, we have

$$\mathcal{V}(w,\Box(\phi\to\psi))\odot\mathcal{V}(w,\Box\phi)\leq\inf\{\mathcal{R}(w,v)\Rightarrow\mathcal{V}(v,\psi)\mid v\in\mathcal{W}\} = \mathcal{V}(w,\Box\psi).$$

Now, suppose that $\mathfrak{M}$ is reflexive. Then we have

$$\mathcal{V}(w,\Box\phi) = \inf\{\mathcal{R}(w,v)\Rightarrow\mathcal{V}(v,\phi)\mid v\in\mathcal{W}\}\leq\mathcal{R}(w,w)\Rightarrow\mathcal{V}(w,\phi) = \mathcal{V}(w,\phi).$$

Lastly, suppose that $\mathfrak{M}$ is transitive. Then, we have for a fixed ϕ that

$$\mathcal{R}(w,v) \leq \mathcal{V}(w,\Box\phi) \Rightarrow \mathcal{V}(v,\Box\phi)$$

and therefore

$$\mathcal{V}(w,\Box\phi) \leq \mathcal{R}(w,v) \Rightarrow \mathcal{V}(v,\Box\phi)$$

for any $v \in \mathcal{W}$ as $\mathfrak{M}$ is transitive. Taking the infimum over v on the right, we have

$$\mathcal{V}(w,\Box\phi) \leq \inf\{\mathcal{R}(w,v) \Rightarrow \mathcal{V}(v,\Box\phi) \mid v \in \mathcal{W}\} = \mathcal{V}(w,\Box\Box\phi).$$

□

3. A Completeness Theorem

To approach completeness, we use a similar overall strategy as [2] and translate modal statements into a propositional language extending $\mathcal{L}_0$. For this, we define the language

$$\mathcal{L}_0(X) : \phi ::= \bot \mid x \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

where $x \in X$ for a countably infinite set X. We write $\mathcal{G}(X)$ for the proof system $\mathcal{G}$ defined over the language $\mathcal{L}_0(X)$ and again we write $\Gamma \vdash_{\mathcal{G}(X)} \phi$ for derivability of ϕ under assumptions Γ with $\Gamma\cup\{\phi\} \subseteq \mathcal{L}_0(X)$ and $\vdash_{\mathcal{G}(X)} \phi$ for $\emptyset \vdash_{\mathcal{G}(X)} \phi$. $\mathcal{L}_0(X)$ can be semantically interpreted by considering *evaluation functions* $v : \mathcal{L}_0(X) \to [0,1]$ which satisfy the following conditions:

- $v(\bot) = 0$;
- $v(\phi \wedge \psi) = v(\phi) \odot v(\psi)$;
- $v(\phi \vee \psi) = v(\phi) \oplus v(\psi)$;
- $v(\phi \rightarrow \psi) = v(\phi) \Rightarrow v(\psi)$.

We write $v[\Gamma]$ for the image set of $\Gamma \subseteq \mathcal{L}_0(X)$ under v. Note, that these functions are uniquely determined by their values on X by recursion on $\mathcal{L}_0(X)$. We denote the set of all such evaluations by $\mathsf{Ev}(\mathcal{L}_0(X))$ and we can use them to define the following notion of semantic consequence which gives a semantical characterization of $\mathcal{G}(X)$.

Definition 8. Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$. Then we write $\Gamma \Vdash \phi$ for $\forall v \in \mathsf{Ev}(\mathcal{L}_0(X))\,(v[\Gamma] \subseteq \{1\}$ implies $v(\phi) = 1)$.

We have the following completeness theorem which was originally obtain by Dummett in [3], using a slightly different semantics (see also [1, 8, 14]).

Theorem 9. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$*:* $\Gamma \vdash_{\mathcal{G}(X)} \phi$ *iff* $\Gamma \Vdash \phi$*.*

In the following, a particular choice for the set of variables X will be

$$Var^\star := Var \cup \{\phi_\Box \mid \phi \in \mathcal{L}_\Box\}$$

and we write $\mathcal{L}_0^\star := \mathcal{L}_0(Var^\star)$ and $\mathcal{G}^\star$ for $\mathcal{G}(Var^\star)$. From $\mathcal{L}_\Box$ to $\mathcal{L}_0^\star$, we define the following translation $\star$ by recursion on $\mathcal{L}_\Box$:

- $\bot^\star := \bot$;
- $p^\star := p$;
- $(\phi \wedge \psi)^\star := \phi^\star \wedge \psi^\star$;
- $(\phi \vee \psi)^\star := \phi^\star \vee \psi^\star$;
- $(\phi \rightarrow \psi)^\star := \phi^\star \rightarrow \psi^\star$;
- $(\Box\phi)^\star := \phi_\Box$.

Extending the translation to sets, we write $[\Gamma]^\star := \{\gamma^\star \mid \gamma \in \Gamma\}$. It is straightforward to see that $\star$ is a bijection and we thus write $\phi^\star$ with $\phi \in \mathcal{L}_\Box$ to denote elements of $\mathcal{L}_0^\star$. We obtain the following lemma, showing that $\star$ is a proof interpretation characterizing provability in $\mathcal{GML}_\Box^-$ in a suitable extension of $\mathcal{G}^\star$. For this, we write $Th_{\mathcal{GML}_\Box^-} := \{\phi \in \mathcal{L}_\Box \mid \vdash_{\mathcal{GML}_\Box^-} \phi\}$.

Lemma 10. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$*:* $\Gamma \vdash_{\mathcal{GML}_\Box^-} \phi$ *iff* $[\Gamma]^\star \cup [Th_{\mathcal{GML}_\Box^-}]^\star \vdash_{\mathcal{G}^\star} \phi^\star$*.*

For a (detailed) proof, see either [2] where the theorem is proved in the context of the standard Gödel modal logics (the proof transfers almost directly) or see [12]. We construct the following canonical model $\mathfrak{M}^c_{\mathcal{GML}_\Box^-}$:

Definition 11. We define $\mathfrak{M}^c_{\mathcal{GML}_\Box^-} := \langle \mathcal{W}^c, \mathcal{W}_0^c, \mathcal{R}^c, \mathcal{V}^c\rangle$ as follows:

(1) $\mathcal{W}^c := \{v \in [0,1]^{\mathcal{L}_0^\star} \mid v[[Th_{\mathcal{GML}_\Box^-}]^\star] \subseteq \{1\}\}$;
(2) $\mathcal{W}_0^c := \{v \in \mathsf{Ev}(\mathcal{L}_0^\star) \mid v[[Th_{\mathcal{GML}_\Box^-}]^\star] \subseteq \{1\}\}$;

(3) $\mathcal{R}^c(v,w) := \begin{cases} 1 & \text{if } \forall\phi\in\mathcal{L}_\Box\,(v(\phi_\Box)\leq w(\phi^\star)) \text{ and } \forall\psi\in\mathcal{L}_\Box\left(\sup_{\phi\in\mathcal{L}_\Box}\{w((\phi\to\psi)^\star)\odot w(\phi^\star)\}\leq w(\psi^\star)\right); \\ 0 & \text{else;}\end{cases}$

(4) $\mathcal{V}^c(v,\phi) := v(\phi^\star)$.

The model is actually well-defined as $\mathcal{W}^c_0$ (and thus $\mathcal{W}^c$) is non-empty. This follows easily from the completeness theorem of [2] which establishes $\not\vdash_{\mathcal{GML}^-_\Box}\bot$.[2] Therefore, we have $[Th_{\mathcal{GML}^-_\Box}]^\star\not\vdash_{\mathcal{G}^\star}\bot$ by Lemma 10. Theorem 9 gives that there exists a $v\in\mathsf{Ev}(\mathcal{L}^\star_0)$ with $v[[Th_{\mathcal{GML}^-_\Box}]^\star]\subseteq\{1\}$, i.e. $v\in\mathcal{W}^c_0$.

Lemma 12. *$\mathfrak{M}^c_{\mathcal{GML}^-_\Box}$ is a well-defined GS-model. Further:*

(1) if (T) is an axiom scheme of $\mathcal{GML}^-_\Box$, then $\mathcal{R}^c$ is reflexive;
(2) if (4) is an axiom scheme of $\mathcal{GML}^-_\Box$, then $\mathcal{R}^c$ is transitive.

Proof. We at first need to verify the conditions (i) - (v) from Definition 3. For this, let $v\in\mathcal{W}^c_0$. We only verify condition (v) as the others are fairly obvious from $v\in\mathsf{Ev}(\mathcal{L}^\star_0)$ alone.

For this, let $\phi\in\mathcal{L}_\Box$. We have by definition that

$$\inf\{\mathcal{R}^c(v,w)\Rightarrow\mathcal{V}^c(w,\phi)\mid w\in\mathcal{W}^c\}=\inf\{w(\phi^\star)\mid w\in\mathcal{W}^c,\mathcal{R}^c(v,w)=1\}$$

where we assume $\inf\emptyset:=1$. Now, for any $w\in\mathcal{W}^c$ with $\mathcal{R}^c(v,w)=1$ we especially have

$$v(\phi_\Box)\leq w(\phi^\star).$$

by definition. Thus, we have

$$\mathcal{V}^c(v,\Box\phi)=v(\phi_\Box)\leq\inf\{w(\phi^\star)\mid w\in\mathcal{W}^c,\mathcal{R}^c(v,w)=1\}.$$

For the converse inequality, we define $v_\Box:\mathcal{L}^\star_0\to[0,1]$ by

$$v_\Box:\phi^\star\mapsto v(\phi_\Box).$$

We need to verify that $v_\Box$ is in $\mathcal{W}^c$. For this, let $\psi^\star\in[Th_{\mathcal{GML}^-_\Box}]^\star$, then $\vdash_{\mathcal{GML}^-_\Box}\psi$ by Lemma 10 and thus $\vdash_{\mathcal{GML}^-_\Box}\Box\psi$ by $(R\Box)$. Therefore $\psi_\Box\in[Th_{\mathcal{GML}^-_\Box}]^\star$ and therefore $v(\psi_\Box)=1$ and thus $v_\Box(\psi^\star)=v(\psi_\Box)=1$.

Further, we have $\mathcal{R}^c(v,v_\Box)=1$ as for one

$$v(\psi_\Box)\leq v_\Box(\psi^\star)=v(\psi_\Box)$$

for all $\psi\in\mathcal{L}_\Box$. For another, note first that we have $v[[Th_{\mathcal{GML}^-_\Box}]^\star]\subseteq\{1\}$ as $v\in\mathcal{W}^c_0$. Thus, we have

$$\begin{aligned} v_\Box((\chi\to\psi)^\star)\odot v_\Box(\chi^\star) &= v((\chi\to\psi)_\Box)\odot v(\chi_\Box)\\ &\leq v(\psi_\Box)=v_\Box(\psi^\star)\end{aligned}$$

for any $\chi\in\mathcal{L}_\Box$ as v satisfies $[(K)]^\star$ and $v\in\mathsf{Ev}(\mathcal{L}^\star_0)$. Thus, we have

$$\sup_{\chi\in\mathcal{L}_\Box}\{v_\Box((\chi\to\psi)^\star)\odot v_\Box(\chi^\star)\}\leq v_\Box(\psi^\star)$$

which implies $\mathcal{R}^c(v,v_\Box)=1$. It follows that

$$\begin{aligned}\inf\{\mathcal{R}^c(v,w)\Rightarrow\mathcal{V}^c(w,\phi)\mid w\in\mathcal{W}^c\} &\leq\mathcal{R}^c(v,v_\Box)\Rightarrow\mathcal{V}^c(v_\Box,\phi)\\ &=1\Rightarrow v(\phi_\Box)=v(\phi_\Box)\end{aligned}$$

and therefore $\mathcal{V}^c(v,\Box\phi)=v(\phi_\Box)=\inf\{\mathcal{R}^c(v,w)\Rightarrow\mathcal{V}^c(w,\phi)\mid w\in\mathcal{W}^c\}$.

We need to verify that $\mathfrak{M}^c_{\mathcal{GML}^-_\Box}$ is regular. For property (b), let first ϕ be such that

$$w(\phi^\star)=\mathcal{V}^c(w,\phi)=1$$

for all $w\in\mathcal{W}^c_0$. Therefore, we have

$$[Th_{\mathcal{GML}^-_\Box}]^\star\vdash_{\mathcal{G}^\star}\phi^\star$$

by Theorem 9 and by Lemma 10, we have $\vdash_{\mathcal{GML}^-_\Box}\phi$, thus $\phi^\star\in[Th_{\mathcal{GML}^-_\Box}]^\star$ and therefore

$$\mathcal{V}^c(w,\phi)=w(\phi^\star)=1$$

for all $w\in\mathcal{W}^c$ by definition.

Further, for property (a), let $\mathcal{R}^c(v,w)=1$. Then, especially

$$\forall\psi\in\mathcal{L}_\Box:\sup_{\phi\in\mathcal{L}_\Box}\{w((\phi\to\psi)^\star)\odot w(\phi^\star)\}\leq w(\psi^\star)$$

[2]Actually, [2] only establishes this implicitly. They actually show $\not\vdash_{\mathcal{GML}_\Box}\bot$ where $\mathcal{GML}_\Box$ is an extension of $\mathcal{GML}^-_\Box$ (see Section 4).

and therefore

$$\inf_{\psi \in \mathcal{L}_\Box} \{ \sup_{\phi \in \mathcal{L}_\Box} \{ w((\phi \to \psi)^\star) \odot w(\phi^\star) \} \Rightarrow w(\psi^\star) \} = 1.$$

Lastly, we verify the two extra conditions (1) and (2).

(1) If (T) is an axiom scheme of $\mathcal{GML}^-_\Box$, then $\vdash_{\mathcal{GML}^-_\Box} \Box\phi \to \phi$ for all $\phi \in \mathcal{L}_\Box$ and thus we have

$$v(\phi_\Box) \Rightarrow v(\phi^\star) = 1$$

for any $\phi \in \mathcal{L}_\Box$ and any $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$ such that $v[[Th_{\mathcal{GML}^-_\Box}]^\star] \subseteq \{1\}$, i.e. for any $v \in \mathcal{W}^c_0$. Thus, we have

$$v(\phi_\Box) \leq v(\phi^\star)$$

for any $\phi \in \mathcal{L}_\Box$ and thus $\mathcal{R}^c(v, v) = 1$ as naturally

$$v(\phi^\star \to \psi^\star) \odot v(\phi^\star) \leq v(\psi^\star)$$

for any $\phi, \psi \in \mathcal{L}_\Box$ through $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$.

(2) For (4) being an axiom scheme of $\mathcal{GML}^-_\Box$, we have $\vdash_{\mathcal{GML}^-_\Box} \Box\phi \to \Box\Box\phi$ for all $\phi \in \mathcal{L}_\Box$ and thus we have

$$v(\phi_\Box) \leq v((\Box\phi)_\Box)$$

for any $\phi \in \mathcal{L}_\Box$ and any $v \in \mathcal{W}^c_0$. Let $w \in \mathcal{W}^c$ and suppose $\mathcal{R}^c(v, w) = 1$. Given $\phi \in \mathcal{L}_\Box$, we have

$$v(\phi_\Box) \leq v((\Box\phi)_\Box) \leq w(\phi_\Box)$$

and therefore $\mathcal{V}^c(v, \Box\phi) \leq \mathcal{V}^c(w, \Box\phi)$. Thus, by taking the infimum over ϕ, we have

$$\inf\{\mathcal{V}^c(v, \Box\phi) \Rightarrow \mathcal{V}^c(w, \Box\phi) \mid \phi \in \mathcal{L}_\Box\} = 1.$$

□

This canonical model construction now culminates in the following completeness theorem.

Theorem 13. *For $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$, the following are equivalent:*

(1) $\Gamma \vdash_{\mathcal{GML}^-_\Box} \phi$;
(2) $\Gamma \models_{\mathsf{GSML}} \phi$;
(3) $\Gamma \models^1_{\mathsf{GSML}} \phi$;
(4) $\Gamma \models^1_{\mathsf{GSML}^c} \phi$.

Proof. We have (1) implies (2) by Lemma 7 and naturally (2) implies (3) and (3) implies (4). For (4) implies (1), suppose $\Gamma \nvdash_{\mathcal{GML}^-_\Box} \phi$. This implies

$$[\Gamma]^\star \cup [Th_{\mathcal{GML}^-_\Box}]^\star \nvdash_{\mathcal{G}^\star} \phi^\star$$

by Lemma 10. Thus, by the completeness theorem (Theorem 9) of $\mathcal{G}^\star$, we have that there exists a $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$ such that

$$v[[\Gamma]^\star] \subseteq \{1\}, v[[Th_{\mathcal{GML}^-_\Box}]^\star] \subseteq \{1\} \text{ and } v(\phi^\star) < 1.$$

By Lemma 12, $\mathfrak{M}^c_{\mathcal{GML}^-_\Box}$ is a well-defined GMSL^c-model and by construction $v \in \mathcal{W}^c_0$. We have

$$\mathcal{V}^c(v, \gamma) = v(\gamma^\star) = 1$$

for all $\gamma \in \Gamma$ but $\mathcal{V}^c(v, \phi) = v(\phi^\star) < 1$. Thus we have $\Gamma \not\models^1_{\mathsf{GSML}^c} \phi$ as $\mathfrak{M}^c_{\mathcal{GML}^-_\Box}$ is accessibility-crisp. □

4. The (Z) axiom scheme

The (Z) axiom scheme, that is

$$\neg\neg\Box\phi \to \Box\neg\neg\phi \quad (Z),$$

was introduced in [2] in the axiomatization of the so called Gödel-Kripke models. In fact, the systems $\mathcal{GML}_\Box$ arising from extending $\mathcal{GML}^-_\Box$ by (Z) are the standard Gödel modal logics from [2]. Due to the relation of the weak Gödel modal logics with the Gödel justification logics from [12], it is to see that

$$(\dagger) \qquad \nvdash_{\mathcal{GML}^-_\Box} \neg\neg\Box p \to \Box\neg\neg p \quad (p \in Var).$$

Using the Gödel-Subset models, we can give another proof of (†) by giving a countermodel. A simple version of such a model is given by $\mathfrak{M} = \langle \{a, b, c\}, \{a, b\}, \mathcal{R}, \mathcal{V}\rangle$ where

$$\mathcal{R}(x, y) := \begin{cases} 1 & \text{for } x = y \text{ or } x = a, y = c, \\ 0 & \text{otherwise.} \end{cases}$$

We define $\mathcal{V}(b, \phi)$ by extending $\mathcal{V}(b, \cdot) : p \mapsto 0$ $(p \in Var)$ recursively to $\mathcal{L}_\Box$ as follows:

- $\mathcal{V}(b, \bot) := 0$;
- $\mathcal{V}(b, \phi \wedge \psi) := \mathcal{V}(b, \phi) \odot \mathcal{V}(b, \psi)$;
- $\mathcal{V}(b, \phi \vee \psi) := \mathcal{V}(b, \phi) \oplus \mathcal{V}(b, \psi)$;
- $\mathcal{V}(b, \phi \to \psi) := \mathcal{V}(b, \phi) \Rightarrow \mathcal{V}(b, \psi)$;
- $\mathcal{V}(b, \Box\phi) := \mathcal{V}(b, \phi)$.

Further, we extend $\mathcal{V}(c, \cdot) : p \mapsto 1/2$ ($p \in Var$) recursively to $\mathcal{L}_\Box$ by:

- $\mathcal{V}(c, \bot) := 1/2$;
- $\mathcal{V}(c, \phi \wedge \psi) := \mathcal{V}(c, \phi) \odot \mathcal{V}(c, \psi)$;
- $\mathcal{V}(c, \phi \vee \psi) := \mathcal{V}(c, \phi) \oplus \mathcal{V}(c, \psi)$;
- $\mathcal{V}(c, \phi \to \psi) := \mathcal{V}(c, \phi) \Rightarrow \mathcal{V}(c, \psi)$;
- $\mathcal{V}(c, \Box\phi) := \mathcal{V}(c, \phi)$.

Lastly, for $\mathcal{V}(a, \phi)$, we extend $\mathcal{V}(a, \cdot) : p \mapsto 1$ ($p \in Var$) to $\mathcal{L}_\Box$ as follows:

- $\mathcal{V}(a, \bot) := 0$;
- $\mathcal{V}(a, \phi \wedge \psi) := \mathcal{V}(a, \phi) \odot \mathcal{V}(a, \psi)$;
- $\mathcal{V}(a, \phi \vee \psi) := \mathcal{V}(a, \phi) \oplus \mathcal{V}(a, \psi)$;
- $\mathcal{V}(a, \phi \to \psi) := \mathcal{V}(a, \phi) \Rightarrow \mathcal{V}(a, \psi)$;
- $\mathcal{V}(a, \Box\phi) := \mathcal{V}(a, \phi) \odot \mathcal{V}(c, \phi)$.

Then, $\mathfrak{M}$ is a well-defined reflexive and transitive model. It is obviously reflexive and for transitivity, we only have to verify

$$\mathcal{V}(a, \Box\phi) \leq \mathcal{V}(c, \Box\phi)$$

but this is immediate by definition. Conditions (i) to (v) from Definition 3 are satisfied for a, b by construction. To see that $\mathfrak{M}$ is regular, first note that

$$\mathcal{V}(c, \phi \to \psi) \odot \mathcal{V}(c, \phi) \leq \mathcal{V}(c, \psi)$$

by definition for all $\phi, \psi \in \mathcal{L}_\Box$ as $\mathcal{V}(c, \phi \to \psi) = \mathcal{V}(c, \phi) \Rightarrow \mathcal{V}(c, \psi)$ by construction. To show that $\mathcal{V}(a, \phi) = \mathcal{V}(b, \phi) = 1$ implies $\mathcal{V}(c, \phi) = 1$, we first show the following claim:
<u>Claim:</u> $\mathcal{V}(c, \phi) = \mathcal{V}(b, \phi) \bar{+} 1/2$ where $\bar{+}$ is bounded addition, that is $x \bar{+} y := x + y$ if $x + y \leq 1$ and $x \bar{+} y := 1$ otherwise (where $x, y \in [0, 1]$).
<u>Proof:</u> We give the proof by induction on ϕ. For $\phi = \bot$ or $\phi = p \in Var$, the claim is immediate. Thus, suppose that ϕ, ψ are formulae that fulfill the claim, that is $\mathcal{V}(c, \phi) = \mathcal{V}(b, \phi) \bar{+} 1/2$ and $\mathcal{V}(c, \psi) = \mathcal{V}(b, \psi) \bar{+} 1/2$. The cases of $\phi \wedge \psi$ and $\phi \vee \psi$ are immediate. We only consider the cases of $\phi \to \psi$ and $\Box\phi$.

The latter follows through

$$\mathcal{V}(c, \Box\phi) = \mathcal{V}(c, \phi) = \mathcal{V}(b, \phi) \bar{+} 1/2 = \mathcal{V}(b, \Box\phi) \bar{+} 1/2.$$

For the former, we have

$$\begin{aligned}
\mathcal{V}(c, \phi \to \psi) &= \mathcal{V}(c, \phi) \Rightarrow \mathcal{V}(c, \psi) \\
&= \left(\mathcal{V}(b, \phi) \bar{+} \frac{1}{2}\right) \Rightarrow \left(\mathcal{V}(b, \psi) \bar{+} \frac{1}{2}\right) \\
&= (\mathcal{V}(b, \phi) \Rightarrow \mathcal{V}(b, \psi)) \bar{+} \frac{1}{2}
\end{aligned}$$

where we have used $(x \bar{+} a) \Rightarrow (y \bar{+} a) = (x \Rightarrow y) \bar{+} a$ with $x, y, a \in [0, 1]$. To see this equality, note that if $x \leq y$, i.e. $x \Rightarrow y = 1$, then $x \bar{+} a \leq y \bar{+} a$, i.e. $x \bar{+} a \Rightarrow y \bar{+} a = 1$. Otherwise, we have $x > y$, i.e. $(x \Rightarrow y) \bar{+} a = y \bar{+} a$ and $x \bar{+} a \geq y \bar{+} a$ where equality occurs only if $y \bar{+} a = 1$. Thus, in effect $x \bar{+} a \Rightarrow y \bar{+} a = y \bar{+} a$. ■

The claim gives especially that $\mathcal{V}(b, \phi) = 1$ implies $\mathcal{V}(c, \phi) = 1$. Now, evaluating (Z) in $\mathfrak{M}$, we get at first

$$\begin{aligned}
\mathcal{V}(a, \neg\neg\Box p) &= \sim^2 \mathcal{V}(a, \Box p) \\
&= \sim^2 (\mathcal{V}(a, p) \odot \mathcal{V}(c, p)) \\
&= \sim^2 (1 \odot 1/2) \\
&= 1.
\end{aligned}$$

However, we have

$$\begin{aligned}\mathcal{V}(c,\neg\neg p) &= (\mathcal{V}(c,p) \Rightarrow \mathcal{V}(c,\bot)) \Rightarrow \mathcal{V}(c,\bot)\\ &= (1/2 \Rightarrow 1/2) \Rightarrow 1/2\\ &= 1 \Rightarrow 1/2\\ &= 1/2\end{aligned}$$

and therefore we obtain

$$\mathcal{V}(a,\Box\neg\neg p) = \mathcal{V}(a,\neg\neg p) \odot \mathcal{V}(c,\neg\neg p) = 1 \odot 1/2 = 1/2.$$

Thus $\mathcal{V}(a,\neg\neg\Box p \to \Box\neg\neg p) = 1/2 < 1$ and by Theorem 13, we have (†).

Through the completeness theorem for $\mathcal{GML}_\Box$ from [2], we get that for any GSML-model $\mathfrak{M}$ with $\mathcal{D}(\mathfrak{M}) = \mathcal{D}_0(\mathfrak{M})$: $\mathfrak{M} \models (Z)$ and further that $\mathcal{GML}_\Box$ is complete with respect to this model class. It may be however interesting to see whether there are other conditions (e.g. on $\mathcal{R}$ or $\mathcal{V}$) for GSML-models which classify a type of models with respect to which $\mathcal{GML}_\Box$ is complete (or (Z) is valid). This may further clarify the role of the (Z)-axiom scheme in the context of the (weak) Gödel modal logics.

We give a particular example of such a condition, more general than $\mathcal{D}(\mathfrak{M}) = \mathcal{D}_0(\mathfrak{M})$, in the following and give a completeness theorem of $\mathcal{GML}_\Box$ with respect to these models.

Definition 14. A Gödel-Subset model $\mathfrak{M} = \langle \mathcal{W}, \mathcal{W}_0, \mathcal{R}, \mathcal{V}\rangle$ is called $\sim^2$*-monotone* if for all $w \in \mathcal{W}$ and all $\phi \in \mathcal{L}_\Box$:

$$\sim^2 \mathcal{V}(w,\phi) \leq \mathcal{V}(w,\neg\neg\phi).$$

Given a class C of GS-models, we denote the subclass of all $\sim^2$-monotone models in C by MC.

Note, that the condition for $\sim^2$-monotonicity is only really required for $w \in \mathcal{W}\setminus\mathcal{W}_0$ as naturally for $w \in \mathcal{W}_0$:

$$\sim^2 \mathcal{V}(w,\phi) = \mathcal{V}(w,\neg\neg\phi).$$

We immediately have the following lemma.

Lemma 15. *For any* MGS*-model* $\mathfrak{M} = \langle \mathcal{W}, \mathcal{W}_0, \mathcal{R}, \mathcal{V}\rangle$*, any* $w \in \mathcal{W}_0$ *and any* $\phi \in \mathcal{L}_\Box$*:*

$$(\mathfrak{M}, w) \models \neg\neg\Box\phi \to \Box\neg\neg\phi.$$

Proof. Let $\mathfrak{M}$ be as required. We have, as $w \in \mathcal{W}_0$, for any $v \in \mathcal{W}$:

$$\begin{aligned}\mathcal{V}(w,\neg\neg\Box\phi) &=\sim^2 \inf\{\mathcal{R}(w,u) \Rightarrow \mathcal{V}(u,\phi) \mid u \in \mathcal{W}\}\\ &\leq\sim^2 (\mathcal{R}(w,v) \Rightarrow \mathcal{V}(v,\phi))\\ &\leq \mathcal{R}(w,v) \Rightarrow\sim^2 \mathcal{V}(v,\phi)\\ &\leq \mathcal{R}(w,v) \Rightarrow \mathcal{V}(v,\neg\neg\phi).\end{aligned}$$

Therefore, by taking the meet over v, we have

$$\mathcal{V}(w,\neg\neg\Box\phi) \leq \inf\{\mathcal{R}(w,v) \Rightarrow \mathcal{V}(v,\neg\neg\phi) \mid v \in \mathcal{W}\} = \mathcal{V}(w,\Box\neg\neg\phi).$$

□

Naturally, Lemma 15 implies the following soundness result.

Lemma 16. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$*, we have* $\Gamma \vdash_{\mathcal{GML}_\Box} \phi$ *implies* $\Gamma \models_{\mathsf{MGSML}} \phi$*.*

We additionally get the converse of the above soundness result, that is completeness of $\mathcal{GML}_\Box$ with respect to MGSML, by the following argument which modifies the previous completeness proof for $\mathcal{GML}_\Box^-$ and GSML.

The idea (although not applied for the canonical model $\mathfrak{M}^c_{\mathcal{GML}_\Box^-}$ already) is that the total set of worlds $\mathcal{W}^c$ can be restricted to $\mathcal{W}_0^c$ together with all $v_\Box$ for $v \in \mathcal{W}_0^c$ where $v_\Box$ is defined as in the proof of Lemma 12. This then gives better control over the worlds $v \in \mathcal{W}^c \setminus \mathcal{W}_0^c$.

At first, as before, we mention the translation lemma for $\star$, now in the context of $\mathcal{GML}_\Box$.

Lemma 17. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$*:* $\Gamma \vdash_{\mathcal{GML}_\Box} \phi$ *iff* $[\Gamma]^\star \cup [Th_{\mathcal{GML}_\Box}]^\star \vdash_{\mathcal{G}^\star} \phi^\star$*.*

In comparison to Lemma 10, this is the version considered in [2]. The canonical model is then defined as follows.

Definition 18. We define $\mathfrak{M}^c_{\mathcal{GML}_\Box} := \langle \mathcal{W}^c, \mathcal{W}^c_0, \mathcal{R}^c, \mathcal{V}^c\rangle$ as follows:

(1) $\mathcal{W}^c_0 := \{v \in \mathsf{Ev}(\mathcal{L}^\star_0) \mid v[[Th_{\mathcal{GML}_\Box}]^\star] \subseteq \{1\}\}$;

(2) $\mathcal{W}^c := \mathcal{W}_0^c \cup \{v_\Box \mid v \in \mathcal{W}_0^c\}$ where $v_\Box : \mathcal{L}_0^\star \to [0,1]$ with $v_\Box : \phi^\star \mapsto v(\phi_\Box)$ for $\phi \in \mathcal{L}_\Box$;

(3) $\mathcal{R}^c(v,w) := \begin{cases} 1 & \text{if } \forall \phi \in \mathcal{L}_\Box \, (v(\phi_\Box) \leq w(\phi^\star)) \, ; \\ 0 & \text{else;} \end{cases}$

(4) $\mathcal{V}^c(v,\phi) = v(\phi^\star)$.

Note, that the previous additional condition on $\mathcal{R}^c$ is superfluous now as every $v_\Box$ naturally adheres to it through the axiom scheme (K).

We obtain a similar results for well-definedness of $\mathfrak{M}^c_{\mathcal{GML}_\Box}$ as in Lemma 12.

Lemma 19. *$\mathfrak{M}^c_{\mathcal{GML}_\Box}$ is a well-defined* MGS*-model. Further*

(1) if (T) is an axiom scheme of $\mathcal{GML}_\Box^-$, then $\mathcal{R}^c$ is reflexive;
(2) if (4) is an axiom scheme of $\mathcal{GML}_\Box^-$, then $\mathcal{R}^c$ is transitive.

Proof. We only show that $\mathfrak{M}^c_{\mathcal{GML}_\Box}$ is $\sim^2$-monotone. The other properties follow as in Lemma 12.

As mentioned before, we only have to consider $w \in \mathcal{W}^c \setminus \mathcal{W}_0^c$. By definition, we have $w = v_\Box$ for some $v \in \mathcal{W}_0^c$. Therefore, we have

$$\begin{aligned} \sim^2 \mathcal{V}^c(v_\Box, \phi) &= \sim^2 v_\Box(\phi^\star) \\ &= \sim^2 v(\phi_\Box) \\ &\leq v((\neg\neg\phi)_\Box) \\ &= v_\Box(\neg\neg\phi^\star) \\ &= \mathcal{V}^c(v_\Box, \neg\neg\phi) \end{aligned}$$

as we have

$$\sim^2 v(\phi_\Box) \leq v((\neg\neg\phi)_\Box)$$

by $\vdash_{\mathcal{GML}_\Box} (Z)$, $v[[Th_{\mathcal{GML}_\Box}]^\star] \subseteq \{1\}$ and $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$. □

As before, the completeness theorem then follows almost per definition of $\mathfrak{M}^c_{\mathcal{GML}_\Box}$.

Theorem 20. *For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$, the following are equivalent:*

(1) $\Gamma \vdash_{\mathcal{GML}_\Box} \phi$;
(2) $\Gamma \models_{\mathsf{MGSML}} \phi$;
(3) $\Gamma \models^1_{\mathsf{MGSML}} \phi$;
(4) $\Gamma \models^1_{\mathsf{MGSML}^c} \phi$.

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