ON INTERMEDIATE JUSTIFICATION LOGICS

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Abstract. We study arbitrary intermediate propositional logics extended with a collection of axioms from (classical) justification logics. For these, we introduce various semantics by combining either Heyting algebras or Kripke frames with the usual semantic machinery used by Mkrtychev's, Fitting's or Lehmann's and Studer's models for classical justification logics. We prove unified completeness theorems for all intermediate justification logics and their corresponding semantics using a respective propositional completeness theorem of the underlying intermediate logic. Further, by a modification of a method of Fitting, we prove unified realization theorems for a large class of intermediate justification logics and accompanying intermediate modal logics.

1. Introduction

Justification logics originated in the 90's from the work of Artemov [1, 2] regarding the provability interpretation of the modal logic **S4** (as initiated by Gödel in [14]) and the connected problem of formalizing the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic propositional logic. From there, the prototype justification logic (the logic of proofs **LP**) was substantially generalized and the resulting family of justification logics gained importance in the context of general (explicit) epistemic reasoning (see the survey [3]) with two recent textbooks on the subject [4, 19].

The original semantics for the logic of proofs was its intended arithmetical interpretation in Peano arithmetic but since then, various other semantics have been proposed which apply not only to the logic of proofs but to the whole family of justification logics. Notable instances important for this paper are the syntactic models of Mkrtychev [23] as well as the possible-world models of Fitting [8] and the recent subset semantics of Lehmann and Studer [20]. These other semantical access points have been instrumental not only in demonstrating the strength of justification logics in modeling general epistemic scenarios and in understanding the ontology of justification terms and formulas in (classical) justification logics but also in inner-logical investigations for properties like decidability (see e.g. [16, 17, 23]).

The main theorem on the logic of proofs, besides arithmetical completeness, is the so-called realization theorem which establishes a correspondence between **S4** and the logic of proofs where every $\Box$ in a modal formula can be (constructively) replaced with a suitable justification term such that the resulting formula is a theorem of **LP**. This property was not only essential to the original motivation of the logic of proofs but is central also in the study of the whole framework of justification logics as it has analogues for all other known classical representatives, giving a central correspondence between justification and modal logics.

Besides the classical justification logics, there is a growing literature on non-classical justification logics, in particular encompassing various lines of research originating from the formalization of explicit, but vague, knowledge. In particular, there are the works on many-valued justification logics (see [11, 12, 27, 28]) and on intuitionistic justification logics (see [18, 21, 22]). In fact, the Gödel justification logics from [11, 27, 28] also relate to the latter as Gödel logic as the base logic is one of the prime examples of an intermediate logic, originating from Dummett's work [7] (in turn influenced by Gödel's remarks on intuitionistic logic [13]).

We give a unified theory regarding semantics and realization for the above examples of intuitionistic, Gödel as well as classical justification logics *and beyond* by introducing abstract intermediate justification logics, that is intermediate propositional logics over the justification language extended with a collection of designated justification axioms.

Semantically, starting at the two typical semantical access points for the underlying intermediate logics of (1) algebraic semantics based on Heyting algebras and of (2) the semantics of Kripke based on partial orders, we extended these algebraic and order theoretic approaches by the usual (appropriately adapted) semantic machinery for treating justification modalities from the classical models of Mkrtychev, Fitting as well as Lehmann and Studer. Here, the algebraic approach extends the classes of classical (or Gödel) Mkrtychev, Fitting and subset

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models by allowing the models to take values not only in $\{0,1\}$ (or $[0,1]$ as in the Gödel case) but in arbitrary (complete) Heyting algebras. The approach via intuitionistic Kripke frames extends the previous considerations for semantics of intuitionistic justification logics.

All these considerations culminate in general unified completeness theorems based on a semantical characterization of the underlying intermediate logic. In particular, we will show that *any* class of algebras or frames complete for the intermediate logic induces a complete class of corresponding models for the justification logic over that base. The proof of this therefore has to rely on different methods than the usual Lindenbaum-Tarski construction.

Concerning realization, we modify the approach of Fitting towards non-constructive classical realization from [9] and prove a very general, though non-constructive, unified realization theorem between these intermediate justification logics and intermediate modal logics. This is a direct example of the applicability of the previous semantic considerations, as this proof of the realization theorem relies on model theoretic constructions using Fitting's models over *intuitionistic* Kripke frames.

2. Intermediate justification logics

2.1. **Syntax and proof calculi.**

We consider the propositional language

$$\mathcal{L}_0 : \phi ::= \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

where $p \in Var := \{p_i \mid i \in \mathbb{N}\}$. We introduce negation via the abbreviation $\neg\phi := \phi \rightarrow \bot$ and write $\top := \neg\bot$. We also define

$$\bigwedge_{i=1}^{n} \phi_n := \phi_1 \wedge \cdots \wedge \phi_n$$

for some $\phi_1, \ldots, \phi_n \in \mathcal{L}_0$. The same applies to $\vee$ and we identify the empty conjunction with $\top$ and the empty disjunction with $\bot$. In order to define intermediate logics and later intermediate justification logics, we need to briefly review some notions regarding propositional substitutions.

A *substitution in* $\mathcal{L}_0$ is a function $\sigma : Var \rightarrow \mathcal{L}_0$. This function σ naturally extends to $\mathcal{L}_0$ by commuting with the connectives $\wedge, \vee, \rightarrow$ and $\bot$ and we write $\sigma(\phi)$ for the image of $\phi \in \mathcal{L}_0$ under this extended function.

Definition 2.1. An *intermediate logic* is a set $\mathbf{L} \subsetneq \mathcal{L}_0$ which satisfies:

(1) the schemes $(A1)$ - $(A9)$ are contained in $\mathbf{L}$;
(2) $\mathbf{L}$ is closed under *modus ponens*, that is $\phi \rightarrow \psi, \phi \in \mathbf{L}$ implies $\psi \in \mathbf{L}$;
(3) $\mathbf{L}$ is closed under substitution in $\mathcal{L}_0$.

Here, the schemes $(A1)$ - $(A9)$ are given by:

$(A1)$ $\phi \rightarrow (\psi \rightarrow \phi)$;
$(A2)$ $(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi))$;
$(A3)$ $(\phi \wedge \psi) \rightarrow \phi$;
$(A4)$ $(\phi \wedge \psi) \rightarrow \psi$;
$(A5)$ $\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$;
$(A6)$ $\phi \rightarrow (\phi \vee \psi)$;
$(A7)$ $\psi \rightarrow (\phi \vee \psi)$;
$(A8)$ $(\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow ((\phi \vee \chi) \rightarrow \psi))$;
$(A9)$ $\bot \rightarrow \phi$.

We denote the smallest intermediate propositional logic, that is the logic given by the axiom schemes $(A1)$ - $(A9)$ in $\mathcal{L}_0$ closed under modus ponens, by **IPC**. Given a set of formulas $\Gamma \subseteq \mathcal{L}_0$, we write

$$\Gamma \vdash_{\mathbf{L}} \phi \text{ iff } \exists \gamma_1, \ldots, \gamma_n \in \Gamma \ \left(\bigwedge_{i=1}^{n} \gamma_i \rightarrow \phi \in \mathbf{L} \right).$$

On the side of justification logics, we consider the following set of *justification terms*

$$Jt : t ::= x \mid c \mid [t + t] \mid [t \cdot t] \mid !t$$

where $x \in V := \{x_i \mid i \in \mathbb{N}\}$ and $c \in C := \{c_i \mid i \in \mathbb{N}\}$ and the resulting multi-modal language

$$\mathcal{L}_J : \phi ::= \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid t : \phi$$

where $p \in Var$ and $t \in Jt$. Naturally, the same abbreviations as for $\mathcal{L}_0$ also apply here. Given sets $\Gamma, \Delta \subseteq \mathcal{L}_J$, we write $\Gamma + \Delta$ for the smallest set containing $\Gamma \cup \Delta$ which is closed under modus ponens.

In order to formulate intermediate justification logics, we need *substitutions in* $\mathcal{L}_J$. Similarly, these are functions $\sigma : Var \rightarrow \mathcal{L}_J$ which extend uniquely to $\mathcal{L}_J$ by commuting with $\wedge, \vee, \rightarrow, \bot$ and the justification modalities 't :'. We again write $\sigma(\phi)$ for the image of a formula $\phi \in \mathcal{L}_J$ under this extension. By $\overline{\Gamma}$, we denote the closure of Γ under substitutions in $\mathcal{L}_J$.

Definition 2.2. Let $\mathbf{L}$ be an intermediate propositional logic. Given the axiom schemes

- (J) $t : (\phi \to \psi) \to (s : \phi \to [t \cdot s] : \psi)$,
- $(+)$ $t : \phi \to [t + s] : \phi$, $t : \phi \to [s + t] : \phi$,
- (F) $t : \phi \to \phi$,
- (I) $t : \phi \to !t : t : \phi$,

we consider the *logic of proofs based on* $\mathbf{L}$:

$$\mathbf{LP}_0(\mathbf{L}) := \overline{\mathbf{L}} + (J) + (+) + (F) + (I).$$

We will focus on this **LP**-case as a particular instructive representative in the realm of different justification logics. All results in this paper naturally extend to intermediate counterparts of the justification logics **J**, **JT** and **J4** defined via

1. $\mathbf{J}_0(\mathbf{L}) := \overline{\mathbf{L}} + (J) + (+)$,
2. $\mathbf{JT}_0(\mathbf{L}) := \overline{\mathbf{L}} + (J) + (+) + (F)$,
3. $\mathbf{J4}_0(\mathbf{L}) := \overline{\mathbf{L}} + (J) + (+) + (I)$,

and probably even to other possible extensions like, e.g., intermediate versions of **J5**.

Given $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, we write

$$\Gamma \vdash_{\mathbf{LP}_0(\mathbf{L})} \phi \text{ iff } \exists \gamma_1, \dots, \gamma_n \in \Gamma \left(\bigwedge_{i=1}^{n} \gamma_i \to \phi \in \mathbf{LP}_0(\mathbf{L}) \right)$$

similar to before.[1]

A *constant specification for* $\mathbf{LP}_0(\mathbf{L})$ is a set CS of formulas from $\mathcal{L}_J$ of the form

$$c_{i_n} : \dots : c_{i_1} : \phi$$

where $n \geq 1$, $c_{i_k} \in C$ for all k and ϕ is an axiom instance of $\mathbf{LP}_0(\mathbf{L})$, that is $\phi \in \overline{\mathbf{L}}$ or ϕ is an instance of the justification axiom schemes $(J), (+), (F), (I)$. We additionally assume that CS is *downwards closed*, i.e. $c_{i_{n+1}} : c_{i_n} : \dots : c_{i_1} : \phi \in CS$ implies $c_{i_n} : \dots : c_{i_1} : \phi \in CS$.[2]

For a given constant specification CS for $\mathbf{LP}_0(\mathbf{L})$, we write $\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$ for $\Gamma \cup CS \vdash_{\mathbf{LP}_0(\mathbf{L})} \phi$ with $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We also write $\mathbf{LP}_{CS}(\mathbf{L}) \vdash \phi$ for $\emptyset \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$.

An important instance of a constant specification for $\mathbf{LP}_0(\mathbf{L})$ is the *total constant specification*, that is the maximal constant specification w.r.t. $\subseteq$ in the sense of the above definition, and we denote it by $TCS_{\mathbf{L}}$. We write $\mathbf{LP}(\mathbf{L})$ for $\mathbf{LP}_{TCS_{\mathbf{L}}}(\mathbf{L})$. The total constant specification will be important later on in the proof of the realization theorems. For this, we already note the following lemma, a straightforward generalization of the classical lifting lemma of justification logics:

Lemma 2.3 (Lifting Lemma). *Let* $\mathbf{L}$ *be an intermediate logic. For* $\{\gamma_1, \dots, \gamma_n, \phi\} \subseteq \mathcal{L}_J$*, if*

$$\{\gamma_1, \dots, \gamma_n\} \vdash_{\mathbf{LP}(\mathbf{L})} \phi,$$

then for any $s_1, \dots, s_n \in Jt$*, there is a* $t \in Jt$ *such that*

$$\{s_1 : \gamma_1, \dots, s_n : \gamma_n\} \vdash_{\mathbf{LP}(\mathbf{L})} t : \phi.$$

A proof for the classical case, which transfers to the intermediate cases immediately, can be found e.g. in [4].

In particular, $\mathbf{LP}(\mathbf{L})$ has *internalization*, that is $\mathbf{LP}(\mathbf{L}) \vdash \phi$ implies that there is a term $t \in Jt$ such that $\mathbf{LP}(\mathbf{L}) \vdash t : \phi$. It should also be noted that the justification variables of t are among the combined justification variables of the s_i. In particular, if the terms s_i do not contain justification variables, then neither does t.

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[1]One could alternatively define $\vdash_{\mathbf{LP}_0(\mathbf{L})}$ via a usual notion of derivation using instances of the axioms schemes $\overline{\mathbf{L}}$, (J), $(+)$, (F), (I) and assumptions together with modus ponens as a rule. We assume this definition implicitly if we prove results by an induction on the length of the derivation.

[2]The downward closure is not needed in the **LP**-case but we include it here to make this definition sound for potential extensions of the results.

2.2. **Extended propositional languages.** In later sections, it will be convenient to consider intermediate logics over different sets of variables. For this, we consider the language

$$\mathcal{L}_0(X) : \phi ::= \bot \mid x \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

with $x \in X$ where X is a countably infinite set of variables. The same notational abbreviations as before also apply here. Note that naturally $\mathcal{L}_0(Var) = \mathcal{L}_0$. A particular choice different from Var for X in the following will be the set

$$Var^\star := Var \cup \{\phi_t \mid \phi \in \mathcal{L}_J, t \in Jt\}.$$

Here, we write $\mathcal{L}_0^\star := \mathcal{L}_0(Var^\star)$.

For the following definition, note that any bijection $t : Var \rightarrow X$ can be naturally extended to a bijection $t : \mathcal{L}_0 \rightarrow \mathcal{L}_0(X)$ through recursion on $\mathcal{L}_0$ by commuting with $\wedge, \vee, \rightarrow$ and $\bot$. Also, such a bijection $t : Var \rightarrow X$ always exists as both X and Var are countably infinite.

Definition 2.4. Let $\mathbf{L}$ be an intermediate logic and let $t : Var \rightarrow X$ be a bijection extended to $t : \mathcal{L}_0 \rightarrow \mathcal{L}_0(X)$ by commuting with $\wedge, \vee, \rightarrow$ and $\bot$. Then we define $\mathbf{L}(X) := t[\mathbf{L}]$.

Note that here also $\mathbf{L}(Var) = \mathbf{L}$.

Remark 2.5. The above definition is invariant under the choice of the bijection $t : Var \rightarrow X$ as $\mathbf{L}$ is closed under substitutions. Further, $\mathbf{L}(X)$ is closed under modus ponens and under substitutions of variables in X by formulas in $\mathcal{L}_0(X)$.

Given $\mathbf{L}(X)$ and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$, we write $\Gamma \vdash_{\mathbf{L}(X)} \phi$ if as before

$$\exists \gamma_1, \dots, \gamma_n \in \Gamma \left(\bigwedge_{i=1}^n \gamma_i \rightarrow \phi \in \mathbf{L}(X) \right).$$

In the following, we will also write $\mathbf{L}^\star$ for the particular case of $\mathbf{L}(Var^\star)$.

3. Algebraic semantics for intermediate justification logics

We move on to the first main line of semantics for intermediate justification logics studied here, extending the model-theoretic approaches of Mkrtychev, Fitting as well as Lehmann and Studer to take values in arbitrary Heyting algebras. The models which we introduce and the techniques used later to prove corresponding completeness theorems are similar to those from [27] where completeness theorems of the particular case of Gödel justification logics with respect to models over the particular Heyting algebra over $[0, 1]$ with the usual order were considered.

3.1. **Heyting algebras and propositional semantics.** We give some preliminaries on Heyting algebras and their relevant notions as a primer for the later definitions.

Definition 3.1. A *Heyting algebra* is a structure $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ such that $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ is a bounded lattice with largest element $1^{\mathbf{A}}$ and smallest element $0^{\mathbf{A}}$ and $\rightarrow^{\mathbf{A}}$ is a binary operation with

(1) $x \rightarrow^{\mathbf{A}} x = 1^{\mathbf{A}}$,
(2) $x \wedge^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} y) = x \wedge^{\mathbf{A}} y$,
(3) $y \wedge^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} y) = y$,
(4) $x \rightarrow^{\mathbf{A}} (y \wedge^{\mathbf{A}} z) = (x \rightarrow^{\mathbf{A}} y) \wedge^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} z)$.

We define $a \leq^{\mathbf{A}} b$ as $a \wedge^{\mathbf{A}} b = a$, which is always a partial order. Given a Heyting algebra $\mathbf{A}$, we write $\neg^{\mathbf{A}} x := x \rightarrow^{\mathbf{A}} 0^{\mathbf{A}}$ (and for reducing parenthesis, we assume that $\neg^{\mathbf{A}}$ binds stronger than the other operations). $\mathbf{A}$ is called a *Boolean algebra* if $x \rightarrow^{\mathbf{A}} y = \neg^{\mathbf{A}} x \vee^{\mathbf{A}} y$ for all $x, y \in A$. For a main reference on basic properties of Heyting algebras, see [29] (and see e.g. [26] for a concise modern one).

A particular property of Heyting algebras important in this note is that of *completeness*.

Definition 3.2. A Heyting algebra $\mathbf{A}$ is *complete* if every set $X \subseteq A$ has a *join* and a *meet* with respect to $\leq^{\mathbf{A}}$, that is for every $X \subseteq A$ there are $s_X, i_X \in A$ such that:

- $\forall x \in X \left(x \leq^{\mathbf{A}} s_X\right)$ and if $x \leq^{\mathbf{A}} s$ for all $x \in X$, then $s_X \leq^{\mathbf{A}} s$;
- $\forall x \in X \left(i_X \leq^{\mathbf{A}} x\right)$ and if $i \leq^{\mathbf{A}} x$ for all $x \in X$, then $i \leq^{\mathbf{A}} i_X$.

We denote these (unique) joins and meets, s_X and i_X, by $\bigvee X$ and $\bigwedge X$, respectively. Given a class of Heyting algebras C, we write C_{com} for the subclass of all complete Heyting algebras in C.

Given an (extended) propositional language $\mathcal{L}_0(X)$, we can give an algebraic interpretation via evaluations into Heyting algebras.

Definition 3.3. Let $\mathbf{A}$ be a Heyting algebra. A *propositional evaluation of* $\mathcal{L}_0(X)$ is a function $f : \mathcal{L}_0(X) \to A$ which satisfies the following equations for all $\phi, \psi \in \mathcal{L}_0(X)$:

(1) $f(\bot) = 0^{\mathbf{A}}$;
(2) $f(\phi \wedge \psi) = f(\phi) \wedge^{\mathbf{A}} f(\psi)$;
(3) $f(\phi \vee \psi) = f(\phi) \vee^{\mathbf{A}} f(\psi)$;
(4) $f(\phi \to \psi) = f(\phi) \to^{\mathbf{A}} f(\psi)$.

We denote the set of all $\mathbf{A}$-valued propositional evaluations of $\mathcal{L}_0(X)$ by $\mathsf{Ev}(\mathbf{A}; \mathcal{L}_0(X))$.

Definition 3.4. Let C be a class of Heyting algebras and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$. We write $\Gamma \models_{\mathsf{C}} \phi$ if

$$\forall \mathbf{A} \in \mathsf{C} \forall f \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0(X)) \big(f[\Gamma] \subseteq \{1^{\mathbf{A}}\} \text{ implies } f(\phi) = 1^{\mathbf{A}}\big).$$

Definition 3.5. Let $\mathbf{L}$ be an intermediate logic. We say that $\mathbf{L}(X)$ is *(strongly) complete* with respect to a class C of Heyting algebras if for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$: $\Gamma \vdash_{\mathbf{L}(X)} \phi$ iff $\Gamma \models_{\mathsf{C}} \phi$.

As is well known (see again e.g. [26]), every intermediate logic actually has at least one class of Heyting algebras with respect to which it is strongly complete (namely its variety). The concrete class will not be of importance in the following considerations and in that way, we introduce the following notation: we write $\mathsf{C} \in \mathsf{Alg}(\mathbf{L}(X))$ or $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L}(X))$ if C is a class of Heyting algebras or of complete Heyting algebras, respectively, with respect to which $\mathbf{L}(X)$ is strongly complete. Note that here

$$\mathsf{C} \in \mathsf{Alg}(\mathbf{L}(X)) \text{ iff } \mathsf{C} \in \mathsf{Alg}(\mathbf{L}(Y))$$

for arbitrary countable sets of variables X, Y and similarly for $\mathsf{Alg}_{com}(\mathbf{L}(X))$.

We now move on to the actual first semantics for the intermediate versions of the logic of proofs.

3.2. Algebraic Mkrtychev models.

The first kind of semantics which we consider are algebraic Mkrtychev models. The classical Mkrtychev models were introduced in [23], originally for the logic of proofs, and mark the first non-provability semantics. The generalization of the Mkrtychev models to the other classical justification logics like $\mathbf{J}, \mathbf{JT}$ and $\mathbf{J4}$ is due to Kuznets [16]. In some contexts, in particular [4, 19], these models are also called basic models. The following algebraic models also generalize the work on $[0, 1]$-valued Mkrtychev models in [11, 27] for the Gödel justification logics.

Definition 3.6 (Algebraic Mkrtychev model)**.** Let $\mathbf{A}$ be a Heyting algebra. An ($\mathbf{A}$*-valued*) *algebraic Mkrtychev model* is a structure $\mathfrak{M} = \langle \mathbf{A}, \mathcal{V} \rangle$ such that $\mathcal{V} : \mathcal{L}_J \to A$ fulfills

(1) $\mathcal{V}(\bot) = 0^{\mathbf{A}}$,
(2) $\mathcal{V}(\phi \wedge \psi) = \mathcal{V}(\phi) \wedge^{\mathbf{A}} \mathcal{V}(\psi)$,
(3) $\mathcal{V}(\phi \vee \psi) = \mathcal{V}(\phi) \vee^{\mathbf{A}} \mathcal{V}(\psi)$,
(4) $\mathcal{V}(\phi \to \psi) = \mathcal{V}(\phi) \to^{\mathbf{A}} \mathcal{V}(\psi)$,

and such that it satisfies

(i) $\mathcal{V}(t : (\phi \to \psi)) \wedge^{\mathbf{A}} \mathcal{V}(s : \phi) \leq^{\mathbf{A}} \mathcal{V}([t \cdot s] : \psi)$,
(ii) $\mathcal{V}(t : \phi) \vee^{\mathbf{A}} \mathcal{V}(s : \phi) \leq^{\mathbf{A}} \mathcal{V}([t + s] : \phi)$,
(iii) $\mathcal{V}(t : \phi) \leq^{\mathbf{A}} \mathcal{V}(\phi)$ (factivity),
(iv) $\mathcal{V}(t : \phi) \leq^{\mathbf{A}} \mathcal{V}(!t : t : \phi)$ (introspectivity),

for all $t, s \in Jt$ and $\phi, \psi \in \mathcal{L}_J$.

Given a class C of Heyting algebras, CAMLP denotes the class of all $\mathbf{A}$-valued Mkrtychev models for all $\mathbf{A} \in \mathsf{C}$. We write $\mathfrak{M} \models \phi$ if $\mathcal{V}(\phi) = 1^{\mathbf{A}}$ and $\mathfrak{M} \models \Gamma$ if $\mathfrak{M} \models \gamma$ for all $\gamma \in \Gamma$ where $\Gamma \subseteq \mathcal{L}_J$.

Further, we say that an algebraic Mkrtychev model $\mathfrak{M}$ *respects* a constant specification CS if $\mathcal{V}(c : \phi) = 1^{\mathbf{A}}$ for all $c : \phi \in CS$. If C is a class of algebraic Mkrtychev models, then we denote the subclass of all models from C respecting a constant specification CS by C_{CS}.

Definition 3.7. Let C be a class of algebraic Mkrtychev models over complete Heyting algebras and let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We write:

(1) $\Gamma \models_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} = \langle \mathbf{A}, \mathcal{V} \rangle \in \mathsf{C} \left(\bigwedge^{\mathbf{A}} \{\mathcal{V}(\gamma) \mid \gamma \in \Gamma\} \leq^{\mathbf{A}} \mathcal{V}(\phi) \right)$;
(2) $\Gamma \models^1_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} = \langle \mathbf{A}, \mathcal{V} \rangle \in \mathsf{C} \Big(\mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \phi \Big)$.

Lemma 3.8. *Let* $\mathbf{L}$ *be an intermediate logic, let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$ *and let* $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \textit{ implies } \Gamma \models_{\mathsf{CAMLP}_{CS}} \phi.$$

The proof is a straightforward induction on the length of the derivation.

3.3. **Algebraic Fitting models.** The second algebraic semantics which we consider are algebraic Fitting models, derived from the fundamental possible-world semantics of Fitting [8] which combined the earlier work of Mkrtychev on syntactic evaluations with the usual semantics of non-explicit modal logics based on modal Kripke models. As a generalization, we allow the accessibility, evidence and evaluation functions to take values in Heyting algebras. The algebraic Fitting models presented here again generalize the previously introduced many-valued Fitting models from [11, 27] from the context of the Gödel justification logics.

Definition 3.9. Let $\mathbf{A}$ be a complete Heyting algebra. An ($\mathbf{A}$-*valued*) *algebraic Fitting model* is a structure $\mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ with

- $\mathcal{W} \neq \emptyset$,
- $\mathcal{R} : \mathcal{W} \times \mathcal{W} \to A$,
- $\mathcal{E} : \mathcal{W} \times Jt \times \mathcal{L}_J \to A$,
- $\mathcal{V} : \mathcal{W} \times \mathcal{L}_J \to A$,

such that it fulfills the conditions

(1) $\mathcal{V}(w, \bot) = 0^{\mathbf{A}}$,
(2) $\mathcal{V}(w, \phi \wedge \psi) = \mathcal{V}(w, \phi) \wedge^{\mathbf{A}} \mathcal{V}(w, \psi)$,
(3) $\mathcal{V}(w, \phi \vee \psi) = \mathcal{V}(w, \phi) \vee^{\mathbf{A}} \mathcal{V}(w, \psi)$,
(4) $\mathcal{V}(w, \phi \to \psi) = \mathcal{V}(w, \phi) \to^{\mathbf{A}} \mathcal{V}(w, \psi)$,
(5) $\mathcal{V}(w, t : \phi) = \mathcal{E}_w(t, \phi) \wedge^{\mathbf{A}} \bigwedge^{\mathbf{A}} \{\mathcal{R}(w, v) \to^{\mathbf{A}} \mathcal{V}(v, \phi) \mid v \in \mathcal{W}\}$,

as well as

(i) $\mathcal{E}_w(t, \phi \to \psi) \wedge^{\mathbf{A}} \mathcal{E}_w(s, \phi) \leq^{\mathbf{A}} \mathcal{E}_w(t \cdot s, \psi)$,
(ii) $\mathcal{E}_w(t, \phi) \vee^{\mathbf{A}} \mathcal{E}_w(s, \phi) \leq^{\mathbf{A}} \mathcal{E}_w(t + s, \phi)$,
(iii) $\mathcal{R}(w, w) = 1^{\mathbf{A}}$ (reflexivity),
(iv) $\mathcal{R}(w, v) \wedge^{\mathbf{A}} \mathcal{R}(v, u) \leq^{\mathbf{A}} \mathcal{R}(w, u)$ (transitivity),
(v) $\mathcal{E}_w(t, \phi) \wedge^{\mathbf{A}} \mathcal{R}(w, v) \leq^{\mathbf{A}} \mathcal{E}_v(t, \phi)$ (monotonicity),
(vi) $\mathcal{E}_w(t, \phi) \leq^{\mathbf{A}} \mathcal{E}_w(!t, t : \phi)$ (introspectivity),

for all $w, v, u \in \mathcal{W}$, all $t, s \in Jt$ and all $\phi, \psi \in \mathcal{L}_J$.

Given a class C of complete Heyting algebras, CAFLP denotes the class of all $\mathbf{A}$-valued Fitting models for all $\mathbf{A} \in \mathsf{C}$. We write $(\mathfrak{M}, w) \models \phi$ for $\mathcal{V}(w, \phi) = 1^{\mathbf{A}}$ and $(\mathfrak{M}, w) \models \Gamma$ if $(\mathfrak{M}, w) \models \gamma$ for all $\gamma \in \Gamma$.

We call an algebraic Fitting model $\mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ *accessibility-crisp* if $\mathcal{R}(w, v) \in \{0^{\mathbf{A}}, 1^{\mathbf{A}}\}$ for all $w, v \in \mathcal{W}$. By C^c, we denote the class of all accessibility-crisp models in C for some class C of algebraic Fitting models.

Further, we say that $\mathfrak{M}$ *respects a constant specification* CS if $\mathcal{V}(w, c : \phi) = 1^{\mathbf{A}}$ for all $w \in \mathcal{W}$ and all $c : \phi \in CS$. Given a class C of algebraic Fitting models, we denote the subclass of all algebraic Fitting models from a class C respecting a constant specification CS by C_{CS} as before.

Definition 3.10. Let C be a class of algebraic Fitting models over complete Heyting algebras and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We write:

(1) $\Gamma \models_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle \in \mathsf{C} \forall w \in \mathcal{W} \left(\bigwedge^{\mathbf{A}} \{\mathcal{V}(w, \gamma) \mid \gamma \in \Gamma\} \leq^{\mathbf{A}} \mathcal{V}(w, \phi)\right)$;
(2) $\Gamma \models^1_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle \in \mathsf{C} \forall w \in \mathcal{W} \Big((\mathfrak{M}, w) \models \Gamma \Rightarrow (\mathfrak{M}, w) \models \phi \Big)$.

Lemma 3.11. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Let* $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \textit{ implies } \Gamma \models_{\mathsf{CAFLP}_{CS}} \phi.$$

Again, the proof is straightforward induction on the length of the derivation.

3.4. **Algebraic subset models.** The last algebraic semantics which we consider are algebraic generalizations of the subset models for classical justification logic by Lehmann and Studer [20].

Definition 3.12. Let $\mathbf{A}$ be a complete Heyting algebra. An ($\mathbf{A}$-*valued*) *algebraic subset model* is a structure $\mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{W}_0, \mathcal{E}, \mathcal{V}\rangle$ with

- $\mathcal{W} \neq \emptyset$,
- $\emptyset \neq \mathcal{W}_0 \subseteq \mathcal{W}$,
- $\mathcal{E} : Jt \times \mathcal{W} \times \mathcal{W} \to A$,

- $\mathcal{V} : \mathcal{W} \times \mathcal{L}_J \to A$,

such that $\mathcal{V}$ fulfills the conditions

(1) $\mathcal{V}(w, \bot) = 0^{\mathbf{A}}$,
(2) $\mathcal{V}(w, \phi \wedge \psi) = \mathcal{V}(w, \phi) \wedge^{\mathbf{A}} \mathcal{V}(w, \psi)$,
(3) $\mathcal{V}(w, \phi \vee \psi) = \mathcal{V}(w, \phi) \vee^{\mathbf{A}} \mathcal{V}(w, \psi)$,
(4) $\mathcal{V}(w, \phi \to \psi) = \mathcal{V}(w, \phi) \to^{\mathbf{A}} \mathcal{V}(w, \psi)$,
(5) $\mathcal{V}(w, t : \phi) = \bigwedge^{\mathbf{A}} \{\mathcal{E}_t(w, v) \to^{\mathbf{A}} \mathcal{V}(v, \phi) \mid v \in \mathcal{W}\}$,

and such that

(i) $\mathcal{E}_{s+t}(w, v) \leq^{\mathbf{A}} \mathcal{E}_s(w, v) \wedge^{\mathbf{A}} \mathcal{E}_t(w, v)$,
(ii)
$$\mathcal{E}_{s\cdot t}(w, v) \leq^{\mathbf{A}} \bigwedge^{\mathbf{A}} \{\mathfrak{M}^w_{s,t}(\psi) \to^{\mathbf{A}} \mathcal{V}(v, \psi) \mid \psi \in \mathcal{L}_J\}$$
with
$$\mathfrak{M}^w_{s,t}(\psi) := \bigvee^{\mathbf{A}} \{\mathcal{V}(w, s : (\phi \to \psi)) \wedge^{\mathbf{A}} \mathcal{V}(w, t : \phi) \mid \phi \in \mathcal{L}_J\},$$
(iii) $\mathcal{E}_t(w, w) = 1^{\mathbf{A}}$ (reflexivity),
(iv) $\mathcal{E}_{!t}(w, v) \leq^{\mathbf{A}} \bigwedge^{\mathbf{A}} \{\mathcal{V}(w, t : \phi) \to^{\mathbf{A}} \mathcal{V}(v, t : \phi) \mid \phi \in \mathcal{L}_J\}$ (introspectivity),

for all $w \in \mathcal{W}_0$, $v \in \mathcal{W}$, $t, s \in Jt$ and $\phi, \psi \in \mathcal{L}_J$.

Let C be a class of complete Heyting algebras. Then CASLP denotes the class of all $\mathbf{A}$-valued subset models for all $\mathbf{A} \in \mathsf{C}$. We write $(\mathfrak{M}, w) \models \phi$ for $\mathcal{V}(w, \phi) = 1^{\mathbf{A}}$ and $(\mathfrak{M}, w) \models \Gamma$ for $\mathcal{V}(w, \gamma) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma$.

The function $\mathcal{E}$ is actually a straightforward $\mathbf{A}$-valued generalization of the E-function from [20] as it is in fact nothing more than a different representation of the function
$$\mathcal{E} : Jt \times \mathcal{W} \to A^{\mathcal{W}}$$
which maps terms and worlds to *A-valued subsets* of $\mathcal{W}$.

We call an algebraic subset model $\mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{W}_0, \mathcal{E}, \mathcal{V}\rangle$ *accessibility-crisp* if $\mathcal{E}_t(w, v) \in \{0^{\mathbf{A}}, 1^{\mathbf{A}}\}$ for all $t \in Jt$ and all $w, v \in \mathcal{W}_0$. Given a class C of algebraic subset models, we denote the class of all accessibility-crisp models in C by C^{c}.

Further, we say that $\mathfrak{M}$ *respects* a constant specification CS if $\mathcal{V}(w, c : \phi) = 1^{\mathbf{A}}$ for all $c : \phi \in CS$ and all $w \in \mathcal{W}_0$ and given a class C of algebraic subset models, we write C_{CS} for the class of all models from C which respect CS.

As before, there are two natural consequence relations to consider here.

Definition 3.13. Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$ and C be a class of algebraic subset models over complete Heyting algebras. We write

(1) $\Gamma \models_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{W}_0, \mathcal{E}, \mathcal{V}\rangle \in \mathsf{C} \forall w \in \mathcal{W}_0 \left(\bigwedge^{\mathbf{A}} \{\mathcal{V}(w, \gamma) \mid \gamma \in \Gamma\} \leq^{\mathbf{A}} \mathcal{V}(w, \phi)\right)$;
(2) $\Gamma \models^1_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} = \langle \mathbf{A}, \mathcal{W}, \mathcal{W}_0, \mathcal{E}, \mathcal{V}\rangle \in \mathsf{C} \forall w \in \mathcal{W}_0 \Big((\mathfrak{M}, w) \models \Gamma \Rightarrow (\mathfrak{M}, w) \models \phi\Big)$.

Lemma 3.14. *Let $\mathbf{L}$ be an intermediate logic and let CS be a constant specification for $\mathbf{LP}_0(\mathbf{L})$. Let $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L})$. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, we have:*
$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \text{ implies } \Gamma \models_{\mathsf{CASLP}_{CS}} \phi.$$

Also here, we omit the proof as it is a simple induction on the length of the derivation.

4. Completeness for algebraic semantics

To approach completeness, we translate the language $\mathcal{L}_J$ to $\mathcal{L}^\star_0$ by introducing the translation
$$\star : \mathcal{L}_J \to \mathcal{L}^\star_0$$
using recursion on $\mathcal{L}_J$ with the following clauses:

- $\bot^\star := \bot$;
- $p^\star := p$;
- $(\phi \circ \psi)^\star := \phi^\star \circ \psi^\star$ with $\circ \in \{\wedge, \vee, \to\}$;
- $(t : \phi)^\star := \phi_t$.

We write $[\Gamma]^\star := \{\gamma^\star \mid \gamma \in \Gamma\}$ for sets $\Gamma \subseteq \mathcal{L}_J$.

Using the above translation, we can convert formulas containing justification modalities into formulas of $\mathcal{L}_0^\star$ and use semantic results for the intermediate logic in question *over* $\mathcal{L}_0^\star$ to derive results for the corresponding intermediate justification logic. This approach, especially in the context of algebra-valued modal logics, goes back to Caicedo and Rodriguez work [5] (see also [31]) and was previously also applied in the context of many-valued justification logics (see [27]).

For this, the following lemma provides a way to interpret modal systems in extended propositional systems where we write $Th_{\mathbf{LP}_{CS}(\mathbf{L})} := \{\phi \in \mathcal{L}_J \mid \mathbf{LP}_{CS}(\mathbf{L}) \vdash \phi\}$.

Lemma 4.1. *Let* $\mathbf{L}$ *be an intermediate logic and* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \text{ iff } [\Gamma]^\star \cup \left[Th_{\mathbf{LP}_{CS}(\mathbf{L})}\right]^\star \vdash_{\mathbf{L}^\star} \phi^\star.$$

In the following, we fix an $\mathbf{L}$ and a constant specification CS for $\mathbf{LP}_0(\mathbf{L})$.

The approach to the uniform completeness results is now as follows: given $\Gamma \not\vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$, we obtain

$$[\Gamma]^\star \cup [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \not\vdash_{\mathbf{L}^\star} \phi^\star$$

by the above Lemma 4.1 and this yields an evaluation $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0^\star)$ for $\mathbf{A} \in \mathsf{C}$, for a complete class of Heyting algebras C for $\mathbf{L}$, such that $v[\Gamma]^\star \subseteq \{1^{\mathbf{A}}\}$[3] and $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}$ but $v(\phi^\star) <^{\mathbf{A}} 1^{\mathbf{A}}$. Using such an evaluation (or the whole set of such evaluations) as a world (or set of worlds), we construct an associated *canonical* algebraic Mkrtychev, Fitting or subset model over $\mathbf{A}$ such that the evaluation in the models (locally at v) corresponds to v. The property $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}$ will guarantee well-definedness and this will be enough to conclude that ϕ does not semantically follow from Γ (in the respective model classes defined using C).

In particular, this construction does not rely on any feature of the class C other than strong completeness w.r.t. $\mathbf{L}$ and since the algebra of the propositional evaluation is preserved as we do note rely on any Lindenbaum-Tarski construction, we achieve the previously mentioned high degree of uniformity: any strongly complete class of algebras for $\mathbf{L}$ induces a corresponding complete class of models for $\mathbf{LP}_{CS}(\mathbf{L})$.

4.1. Completeness w.r.t. algebraic Mkrtychev models.

Definition 4.2. Let $\mathbf{A}$ be a Heyting algebra and $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0^\star)$. The *canonical algebraic Mkrtychev model w.r.t.* $\mathbf{A}$ *and* v is the structure $\mathfrak{M}^{c,M}_{\mathbf{A},v} := \langle \mathbf{A}, \mathcal{V}^c \rangle$ defined by:

$$\mathcal{V}^c(\phi) := v(\phi^\star).$$

Lemma 4.3. *For any* $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0^\star)$ *with* $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}$*,* $\mathfrak{M}^{c,M}_{\mathbf{A},v}$ *is a well-defined* $\mathbf{A}$*-valued algebraic Mkrtychev model which respects* CS*.*

Proof. As $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0^\star)$, we have items (1) - (4) from Definition 3.6. Then, as additionally $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}$, we get

$$\begin{aligned}\mathcal{V}^c(t : (\phi \to \psi)) \wedge^{\mathbf{A}} \mathcal{V}^c(s : \phi) &= v((\phi \to \psi)_t) \wedge^{\mathbf{A}} v(\phi_s) \\ &\leq^{\mathbf{A}} v(\psi_{[t \cdot s]}) \\ &= \mathcal{V}^c([t \cdot s] : \psi)\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}^c(t : \phi) \vee^{\mathbf{A}} \mathcal{V}^c(s : \phi) &= v(\phi_t) \vee^{\mathbf{A}} v(\phi_s) \\ &\leq^{\mathbf{A}} v(\phi_{[t+s]}) \\ &= \mathcal{V}^c([t + s] : \phi)\end{aligned}$$

regarding items (i) and (ii) of Definition 3.6. Now as (F) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, we obtain

$$\mathcal{V}^c(t : \phi) = v(\phi_t) \leq^{\mathbf{A}} v(\phi^\star) = \mathcal{V}^c(\phi).$$

As (I) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, we have

$$\mathcal{V}^c(t : \phi) = v(\phi_t) \leq^{\mathbf{A}} v((t : \phi)_{!t}) = \mathcal{V}^c(!t : t : \phi).$$

□

Following the outline given at the beginning of this section, it is immediately clear how to obtain the following theorem.

[3]We omit the outer brackets of $v[\cdot]$ for better readability.

Theorem 4.4. *Let* $\mathbf{L}$ *be an intermediate logic and* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Further, let* $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*, the following are equivalent:*

(1) $\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$*;*
(2) $\Gamma \models_{\mathsf{CAMLP}_{CS}} \phi$*;*
(3) $\Gamma \models^1_{\mathsf{CAMLP}_{CS}} \phi$*.*

Note that $\mathsf{C} \in \mathsf{Alg}(\mathbf{L})$ suffices if one only wants to establish the equivalence of (1) and (3).

4.2. **Completeness w.r.t. algebraic Fitting models.** While the algebraic Mkrtychev model used a specific evaluation v with $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^\mathbf{A}\}$, the upcoming algebraic Fitting model uses the whole set as a set of worlds and this can even be achieved with a crisp accessibility function.

Definition 4.5. Let $\mathbf{A}$ be a complete Heyting algebra. The *canonical algebraic Fitting model w.r.t.* $\mathbf{A}$ is the structure $\mathfrak{M}^{c,F}_{\mathbf{A}} := \langle \mathbf{A}, \mathcal{W}^c, \mathcal{R}^c, \mathcal{E}^c, \mathcal{V}^c \rangle$ defined as follows:

- $\mathcal{W}^c := \{v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0^\star) \mid v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^\mathbf{A}\}\}$;
- $\mathcal{R}^c(v,w) := \begin{cases} 1^\mathbf{A} & \text{if } \forall t \in Jt \forall \phi \in \mathcal{L}_J \left(v(\phi_t) \leq^\mathbf{A} w(\phi^\star)\right); \\ 0^\mathbf{A} & \text{otherwise;} \end{cases}$
- $\mathcal{E}^c_v(t,\phi) := v(\phi_t)$;
- $\mathcal{V}^c(v,\phi) := v(\phi^\star)$.

Lemma 4.6. *For any complete Heyting algebra* $\mathbf{A}$*,* $\mathfrak{M}^{c,F}_{\mathbf{A}}$ *is a well-defined* $\mathbf{A}$*-valued algebraic Fitting model respecting* CS*.*

Proof. Conditions (1) - (4) from Definition 3.9 follow immediately for any $v \in \mathcal{W}^c$ as $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}_0^\star)$ and by definition of $\star$. For item (5), we have

$$v(\phi_t) \leq^\mathbf{A} w(\phi^\star)$$

for any $w \in \mathcal{W}^c$ with $\mathcal{R}^c(v,w) = 1^\mathbf{A}$. Thus, we obtain

$$v(\phi_t) \leq^\mathbf{A} \bigwedge\nolimits^\mathbf{A} \{w(\phi^\star) \mid w \in \mathcal{W}^c, \mathcal{R}^c(v,w) = 1^\mathbf{A}\} = \bigwedge\nolimits^\mathbf{A} \{\mathcal{R}^c(v,w) \to^\mathbf{A} w(\phi^\star) \mid w \in \mathcal{W}^c\}.$$

Therefore

$$\mathcal{E}^c_v(t,\phi) \wedge^\mathbf{A} \bigwedge\nolimits^\mathbf{A} \{\mathcal{R}^c(v,w) \to^\mathbf{A} w(\phi^\star) \mid w \in \mathcal{W}^c\} = v(\phi_t).$$

Items (i) and (ii) can be shown as in the case of algebraic Mkrtychev models. As (F) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$:

$$v(\phi_t) \leq^\mathbf{A} v(\phi^\star)$$

for any $\phi \in \mathcal{L}_J$ and any $t \in Jt$. Thus, in particular we have $\mathcal{R}^c(v,v) = 1^\mathbf{A}$ by definition and hence $\mathcal{R}^c$ is reflexive.

By the axiom scheme (I), we have

$$\mathcal{E}^c_v(t,\phi) = v(\phi_t) \leq^\mathbf{A} v((t:\phi)_{!t}) = \mathcal{E}^c_v(!t, t:\phi)$$

for any $\phi \in \mathcal{L}_J$ and any $t \in Jt$. Further, we have that $\mathcal{R}^c$ is transitive: let $\mathcal{R}^c(v,w) = \mathcal{R}^c(w,u) = 1^\mathbf{A}$. Then for any $\phi \in \mathcal{L}_J$ and any $t \in Jt$:

$$v(\phi_t) \leq^\mathbf{A} v((t:\phi)_{!t}) \leq^\mathbf{A} w(\phi_t) \leq^\mathbf{A} u(\phi^\star)$$

and thus $\mathcal{R}^c(v,u) = 1^\mathbf{A}$. For the property of monotonicity, suppose $\mathcal{R}^c(v,w) = 1^\mathbf{A}$. Then, we obtain

$$\mathcal{E}^c_v(t,\phi) = v(\phi_t) \leq^\mathbf{A} v((t:\phi)_{!t}) \leq^\mathbf{A} w(\phi_t) = \mathcal{E}^c_w(t,\phi).$$

□

Similarly as before, we obtain the following completeness theorem.

Theorem 4.7. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Further, let* $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*, the following are equivalent:*

(1) $\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$*;*
(2) $\Gamma \models_{\mathsf{CAFLP}_{CS}} \phi$*;*
(3) $\Gamma \models^1_{\mathsf{CAFLP}_{CS}} \phi$*;*
(4) $\Gamma \models^1_{\mathsf{CAFLP}^c{}_{CS}} \phi$*.*

4.3. Completeness w.r.t. algebraic subset models.

Definition 4.8. Let $\mathbf{A}$ be a complete Heyting algebra. The *canonical algebraic subset model w.r.t.* $\mathbf{A}$ is the structure $\mathfrak{M}^{c,S}_{\mathbf{A}} := \langle \mathbf{A}, \mathcal{W}^c, \mathcal{W}^c_0, \mathcal{E}^c, \mathcal{V}^c\rangle$ defined as follows:

- $\mathcal{W}^c := \mathbf{A}^{\mathcal{L}_J}$;
- $\mathcal{W}^c_0 := \{v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}^\star_0) \mid v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}\}$;
- $\mathcal{E}^c_t(v, w) := \begin{cases} 1^{\mathbf{A}} & \text{if } \forall \phi \in \mathcal{L}_J \left(v(\phi_t) \leq^{\mathbf{A}} w(\phi^\star)\right); \\ 0^{\mathbf{A}} & \text{otherwise;} \end{cases}$
- $\mathcal{V}^c(v, \phi) := v(\phi^\star)$.

Lemma 4.9. *For any complete Heyting algebra* $\mathbf{A}$*,* $\mathfrak{M}^{c,S}_{\mathbf{A}}$ *is a well-defined* $\mathbf{A}$*-valued algebraic subset model respecting* CS*.*

Proof. Conditions (1) - (4) follow naturally from $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}^\star_0)$.

For (5), we show the equality in two steps. At first, note that

$$\begin{aligned} \bigwedge\nolimits^{\mathbf{A}}\{\mathcal{E}^c_t(v, w) \to^{\mathbf{A}} \mathcal{V}^c(w, \phi) \mid w \in \mathcal{W}^c\} &= \bigwedge\nolimits^{\mathbf{A}}\{\mathcal{E}^c_t(v, w) \to^{\mathbf{A}} w(\phi^\star) \mid w \in \mathcal{W}^c\} \\ &= \bigwedge\nolimits^{\mathbf{A}}\{w(\phi^\star) \mid w \in \mathcal{W}^c, \mathcal{E}^c_t(v, w) = 1^{\mathbf{A}}\}. \end{aligned}$$

Now, by definition we have

$$\mathcal{V}^c(v, t : \phi) = v(\phi_t) \leq^{\mathbf{A}} w(\phi^\star)$$

for any $w \in \mathcal{W}^c$ with $\mathcal{E}^c_t(v, w) = 1^{\mathbf{A}}$. Thus, we naturally have

$$\mathcal{V}^c(v, t : \phi) \leq^{\mathbf{A}} \bigwedge\nolimits^{\mathbf{A}}\{w(\phi^\star) \mid w \in \mathcal{W}^c, \mathcal{E}^c_t(v, w) = 1^{\mathbf{A}}\}.$$

For the other direction, consider

$$v_t : \mathcal{L}_J \to \mathbf{A}, \psi^\star \mapsto v(\psi_t).$$

Then, we have that $v_t \in \mathcal{W}^c$ and $\mathcal{E}^c_t(v, v_t) = 1^{\mathbf{A}}$ and therefore

$$\bigwedge\nolimits^{\mathbf{A}}\{w(\phi^\star) \mid w \in \mathcal{W}^c, \mathcal{E}^c_t(v, w) = 1^{\mathbf{A}}\} \leq^{\mathbf{A}} v_t(\phi^\star) = v(\phi_t).$$

Let further $w \in \mathcal{W}^c$.

(i) Suppose $\mathcal{E}^c_{t+s}(v, w) = 1^{\mathbf{A}}$. Then, we have (as $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}^\star_0)$ and $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}$)

$$v(\phi_t) \leq^{\mathbf{A}} v(\phi_{[t+s]}) \leq^{\mathbf{A}} w(\phi^\star)$$

through axiom scheme (+) for any $\phi \in \mathcal{L}_J$ and similarly for $v(\phi_s)$. Thus, we have $\mathcal{E}^c_s(v, w) = \mathcal{E}^c_t(v, w) = 1^{\mathbf{A}}$.

(ii) Suppose $\mathcal{E}^c_{t\cdot s}(v, w) = 1^{\mathbf{A}}$. We write $(\mathfrak{M}^c)^v_{t,s}$ as a shorthand for $(\mathfrak{M}^{c,S}_{\mathbf{A}})^v_{t,s}$. Then, to show

$$(\mathfrak{M}^c)^v_{t,s}(\psi) \leq^{\mathbf{A}} w(\psi^\star)$$

for every $\psi \in \mathcal{L}_J$, it suffices to note that

$$\begin{aligned} (\mathfrak{M}^c)^v_{t,s}(\psi) &= \bigvee\nolimits^{\mathbf{A}}\{\mathcal{V}^c(v, t : (\phi \to \psi)) \wedge^{\mathbf{A}} \mathcal{V}^c(v, s : \phi) \mid \phi \in \mathcal{L}_J\} \\ &= \bigvee\nolimits^{\mathbf{A}}\{v((\phi \to \psi)_t) \wedge^{\mathbf{A}} v(\phi_s) \mid \phi \in \mathcal{L}_J\} \\ &\leq^{\mathbf{A}} v(\psi_{t\cdot s}) \end{aligned}$$

for arbitrary $\psi \in \mathcal{L}_J$.

(iii) As (F) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, it follows for any $v \in \mathcal{W}^c_0$ and any $t \in Jt$ that

$$v(\phi_t) \leq^{\mathbf{A}} v(\phi^\star)$$

as $v \in \mathsf{Ev}(\mathbf{A}; \mathcal{L}^\star_0)$ and $v[Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \subseteq \{1^{\mathbf{A}}\}$ again. This gives $\mathcal{E}^c_t(v, v) = 1^{\mathbf{A}}$.

(iv) As (I) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, for arbitrary $v \in \mathcal{W}^c_0$, $w \in \mathcal{W}^c$ and $t \in Jt$ and assuming $\mathcal{E}^c_{!t}(v, w) = 1^{\mathbf{A}}$, we have

$$v(\phi_t) \leq^{\mathbf{A}} v((t : \phi)_{!t})$$

for arbitrary $\phi \in \mathcal{L}_J$ through $v \in \mathcal{W}_0^c$. Therefore

$$\begin{aligned}\mathcal{V}^c(v, t : \phi) &= v(\phi_t)\\ &\leq^{\mathbf{A}} v((t : \phi)_{!t})\\ &\leq^{\mathbf{A}} w(\phi_t)\\ &= \mathcal{V}^c(w, t : \phi)\end{aligned}$$

where the last inequality follows from $\mathcal{E}^c_{!t}(v, w) = 1^{\mathbf{A}}$.

□

Again, following the general outline presented before yields the corresponding completeness result.

Theorem 4.10. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Let further* $\mathsf{C} \in \mathsf{Alg}_{com}(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*, the following are equivalent:*

(1) $\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$*;*
(2) $\Gamma \models_{\mathsf{CASLP}_{CS}} \phi$*;*
(3) $\Gamma \models^1_{\mathsf{CASLP}_{CS}} \phi$*;*
(4) $\Gamma \models^1_{\mathsf{CASLP}^c{}_{CS}} \phi$*.*

5. Frame semantics for intermediate justification logics

As a second semantic approach, we extend not Heyting algebras but intuitionistic Kripke frames for intermediate logics with the semantic machinery of the models of Mkrtychev, Fitting or of Lehmann and Studer.

This extends the work on intuitionistic Mkrtychev and Fitting models (under different terminology) from Marti and Studer in [21] to wider classes of logics.

5.1. **Kripke frames and propositional semantics.**

We review some concepts from Kripke frames for propositional intermediate logics (see e.g. [10, 24]). For this, we need some terminology from order theory.

Definition 5.1. A *Kripke frame* is a structure $\langle F, \leq\rangle$ such that $\leq$ is a partial order on the non-empty set F.

A set $X \subseteq F$ is called a *cone* (or *upset*) if

$$\forall x \in X \forall y \in F\, (x \leq y \Rightarrow y \in X)\,.$$

We denote the smallest cone containing a set X of a partial order $\langle F, \leq\rangle$ by $\uparrow X$. A cone X is called *principal* if $X = \uparrow \{x\}$ for some element x.

A Kripke frame $\mathfrak{G} = \langle G, \leq'\rangle$ is an *(induced) subframe* of a Kripke frame $\mathfrak{F} = \langle F, \leq\rangle$ if $G \subseteq F$ and $\leq' = \leq \cap (G \times G)$. In this case, we also write $\mathfrak{G} = \mathfrak{F} \upharpoonright G$. A Kripke frame is called *principal* if its domain is principal.

Definition 5.2. Let $\mathfrak{F} = \langle F, \leq\rangle$ be a Kripke frame. A ($\mathcal{L}_0(X)$-)*Kripke model* based on $\mathfrak{F}$ is a structure $\mathfrak{M} = \langle \mathfrak{F}, \Vdash\rangle$ with $\Vdash \subseteq F \times X$ which satisfies

$$x \leq y \text{ and } x \Vdash p \text{ implies } y \Vdash p$$

for all $p \in X$.

A Kripke model $\mathfrak{N} = \langle \mathfrak{G}, \Vdash'\rangle$ is called an (*induced*) *submodel* of a Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \Vdash\rangle$ if $\mathfrak{G}$ is an induced subframe of $\mathfrak{F}$ and for all $p \in X$:

$$\{x \in G \mid x \Vdash' p\} = \{x \in F \mid x \Vdash p\} \cap G.$$

We write $\mathfrak{N} = \mathfrak{M} \upharpoonright G$ in this case.

Given a Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \Vdash\rangle$, we introduce the satisfaction relation $\models$ by recursion on the structure of the formula $\phi \in \mathcal{L}_0(X)$:

- $(\mathfrak{M}, x) \not\models \bot$;
- $(\mathfrak{M}, x) \models p$ if $x \Vdash p$;
- $(\mathfrak{M}, x) \models \phi \wedge \psi$ if $(\mathfrak{M}, x) \models \phi$ and $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \vee \psi$ if $(\mathfrak{M}, x) \models \phi$ or $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \rightarrow \psi$ if $\forall y \in F\, (x \leq y \Rightarrow (\mathfrak{M}, y) \not\models \phi$ or $(\mathfrak{M}, y) \models \psi)$.

We write $\mathfrak{M} \models \phi$ if $(\mathfrak{M}, x) \models \phi$ for any $x \in F$, $(\mathfrak{M}, x) \models \Gamma$ if $(\mathfrak{M}, x) \models \gamma$ for all $\gamma \in \Gamma$ and $\mathfrak{M} \models \Gamma$ if $(\mathfrak{M}, x) \models \Gamma$ for all $x \in F$. Further, we write $\mathcal{D}(\mathfrak{M}) := F$.

A fundamental property of Kripke models is that the monotonicity of propositional variables extends to all formulas, i.e. for all $\phi \in \mathcal{L}_0(X)$ and all $x, y \in F$:

$$x \leq y \text{ and } (\mathfrak{M}, x) \models \phi \text{ implies } (\mathfrak{M}, y) \models \phi.$$

The proof is an easy induction on the structure of $\mathcal{L}_0(X)$. Given a class of Kripke frames C, we write $\mathsf{Mod}(\mathsf{C};\mathcal{L}_0(X))$ for the class of all Kripke models over $\mathcal{L}_0(X)$ with underlying Kripke frames from C.

Using these definitions, there are now two definitions of consequence to consider.

Definition 5.3. Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$ and C be a class of Kripke models. Then, we write:

(1) $\Gamma \models_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} \in \mathsf{C} \forall x \in \mathcal{D}(\mathfrak{M})\Big((\mathfrak{M}, x) \models \Gamma \Rightarrow (\mathfrak{M}, x) \models \phi\Big)$;
(2) $\Gamma \models^g_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} \in \mathsf{C}\Big(\mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \phi\Big)$.

Further, if C is now a class of Kripke frames, we write:

(3) $\Gamma \models_{\mathsf{C}} \phi$ if $\Gamma \models_{\mathsf{Mod}(\mathsf{C};\mathcal{L}_0(X))} \phi$;
(4) $\Gamma \models^g_{\mathsf{C}} \phi$ if $\Gamma \models^g_{\mathsf{Mod}(\mathsf{C};\mathcal{L}_0(X))} \phi$.

Definition 5.4. Let $\mathbf{L}$ be an intermediate logic, X a countably infinite set of variables and C be a class of Kripke frames.

(1) We say that $\mathbf{L}(X)$ is *strongly complete* w.r.t. C if $\Gamma \vdash_{\mathbf{L}(X)} \phi$ is equivalent to $\Gamma \models_{\mathsf{C}} \phi$.
(2) We say that $\mathbf{L}(X)$ is *strongly globally complete* w.r.t. C if $\Gamma \vdash_{\mathbf{L}(X)} \phi$ is equivalent to $\Gamma \models^g_{\mathsf{C}} \phi$.

Given a class of Kripke frames C, we write $\mathsf{C} \in \mathsf{KFr}(\mathbf{L})$ or $\mathsf{C} \in \mathsf{KFr}^g(\mathbf{L})$ if $\mathbf{L}$ is strongly (locally) complete or strongly globally complete w.r.t. C, respectively. We also write $\mathsf{C} \in \mathsf{KFr}(\mathbf{L}) \cap \mathsf{KFr}^g(\mathbf{L})$ for $\mathsf{C} \in \mathsf{KFr}(\mathbf{L})$ and $\mathsf{C} \in \mathsf{KFr}^g(\mathbf{L})$.

The global version will later prove to be important in the completeness considerations. Two things shall be noted in this context. First, it is well known that there are Kripke incomplete intermediate logics, that is intermediate logics where there is no class of Kripke frames for which the logic is (even weakly) complete. The first such logic was constructed in [30]. All following considerations involving propositional completeness w.r.t. classes of Kripke frames thus implicitly assume that such a class exists.

Further, if an intermediate logic is characterized by a class of Kripke frames *locally*, there is a simple extended class of frames which characterizes the logic *globally*. More precisely, we have the following:

Lemma 5.5. *Let C be a class of Kripke frames and let $\overline{\mathsf{C}}$ be the closure of C under principal subframes. Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$. Then, we have:*

(1) $\Gamma \models_{\mathsf{C}} \phi$ iff $\Gamma \models_{\overline{\mathsf{C}}} \phi$;
(2) $\Gamma \models_{\overline{\mathsf{C}}} \phi$ iff $\Gamma \models^g_{\overline{\mathsf{C}}} \phi$.

The result is in some way folklore. In any case, the proof is rather immediate, one just restricts to substructures induced by principal subframes, i.e. $\mathfrak{M} \upharpoonright (\uparrow \{x\})$. Over those substructures, local validity in $(\mathfrak{M}, x)$ transfers to global validity in $(\mathfrak{M} \upharpoonright (\uparrow \{x\}), x)$.

5.2. **Intuitionistic Mkrtychev models.** We continue our semantical investigations into intermediate justification logics by extending the approach of Mkrtychev's syntactic models by intuitionistic Kripke frames. These intuitionistic Mkrtychev models are akin to the previously considered models from [21] for $\mathbf{LP}_{CS}(\mathbf{IPC})$ (under the name of intuitionistic basic models).

Definition 5.6 (essentially [21])**.** Let $\mathfrak{F} = \langle F, \leq \rangle$ be a Kripke frame. An *intuitionistic Mkrtychev model based on* $\mathfrak{F}$ is a structure $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{E}, \Vdash \rangle$ such that $\Vdash \subseteq F \times Var$ and $\mathcal{E} : Jt \times F \to \mathcal{P}(\mathcal{L}_J)$ satisfy

(1) $x \leq y$ and $x \Vdash p$ implies $y \Vdash p$,
(2) $x \leq y$ and $\phi \in \mathcal{E}_t(x)$ implies $\phi \in \mathcal{E}_t(y)$,

as well as

(i) $\mathcal{E}_t(x) \sqsupset \mathcal{E}_s(x) \subseteq \mathcal{E}_{[t \cdot s]}(x)$,
(ii) $\mathcal{E}_t(x) \cup \mathcal{E}_s(x) \subseteq \mathcal{E}_{[t+s]}(x)$,
(iii) $\phi \in \mathcal{E}_t(x)$ implies $(\mathfrak{M}, x) \models \phi$ (factivity),
(iv) $t : \mathcal{E}_t(x) \subseteq \mathcal{E}_{!t}(x)$ where $t : \Gamma = \{t : \gamma \mid \gamma \in \Gamma\}$ (introspectivity),

for all $p \in Var$, $\phi \in \mathcal{L}_J$, $t, s \in Jt$ and $x, y \in F$ where

$$\Gamma \sqsupset \Delta := \{\phi \in \mathcal{L}_J \mid \psi \to \phi \in \Gamma, \psi \in \Delta \text{ for some } \psi \in \mathcal{L}_J\}$$

for $\Gamma, \Delta \subseteq \mathcal{L}_J$ and where $\models$ is defined by recursion via

- $(\mathfrak{M}, x) \not\models \bot$;
- $(\mathfrak{M}, x) \models p$ if $x \Vdash p$;
- $(\mathfrak{M}, x) \models \phi \land \psi$ if $(\mathfrak{M}, x) \models \phi$ and $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \lor \psi$ if $(\mathfrak{M}, x) \models \phi$ or $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \to \psi$ if $\forall y \in F\, (x \leq y \Rightarrow (\mathfrak{M}, x) \not\models \phi$ or $(\mathfrak{M}, x) \models \psi)$;

- $(\mathfrak{M}, x) \models t : \phi$ if $\phi \in \mathcal{E}_t(x)$.

Given a class C of Kripke frames, we write CKMLP for the class of all intuitionistic Mkrtychev models over frames from C. Given an intuitionistic Mkrtychev model $\mathfrak{M}$ over a Kripke frame $\mathfrak{F} = \langle F, \leq \rangle$, we also write $\mathcal{D}(\mathfrak{M}) := F$ and call F the domain of $\mathfrak{M}$. We write $(\mathfrak{M}, x) \models \Gamma$ if $(\mathfrak{M}, x) \models \gamma$ for all $\gamma \in \Gamma$.

Further, one can immediately show that the models have the monotonicity property, i.e. for any $\phi \in \mathcal{L}_J$ and all $x, y \in F$:

$$x \leq y \text{ and } (\mathfrak{M}, x) \models \phi \text{ implies } (\mathfrak{M}, y) \models \phi.$$

Given some constant specification CS, we say that $\mathfrak{M}$ *respects* CS if $\phi \in \mathcal{E}_c(x)$ for all $x \in F$ and all $c : \phi \in CS$ and given a class C of intuitionistic Mkrtychev models, we denote the class of all intuitionistic Mkrtychev models from C respecting a constant specification CS by C_{CS}.

Definition 5.7. Let C be a class of intuitionistic Mkrtychev models. We write $\Gamma \models_{\mathsf{C}} \phi$ if for all $\mathfrak{M} \in \mathsf{C}$ and all $x \in \mathcal{D}(\mathfrak{M})$: $(\mathfrak{M}, x) \models \Gamma$ implies $(\mathfrak{M}, x) \models \phi$.

Soundness is then immediate.

Lemma 5.8. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Let* $\mathsf{C} \in \mathsf{KFr}(\mathbf{L})$*. For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \textit{ implies } \Gamma \models_{\mathsf{CKMLP}_{CS}} \phi.$$

5.3. **Intuitionistic Fitting models.** We continue with intuitionistic Fitting models, combining various streams of semantics in non-classical modal logics by either extending the approach via intuitionistic modal Kripke models of [25] for intuitionistic modal logics by the machinery of evidence functions for explicit modalities in the sense of Fitting, or conversely extending Fitting's models with the machinery of intuitionistic Kripke frames. In any way, the models which we introduce are akin to a model class from [21] for $\mathbf{LP}_{CS}(\mathbf{IPC})$ (which are called intuitionistic modular models there).

Definition 5.9 (essentially [21])**.** Let $\mathfrak{F} = \langle F, \leq \rangle$ be a Kripke frame. An *intuitionistic Fitting model based on* $\mathfrak{F}$ is a structure $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{R}, \mathcal{E}, \Vdash \rangle$ such that $\Vdash \subseteq F \times Var$, $\mathcal{R} \subseteq F \times F$ and $\mathcal{E} : Jt \times F \to \mathcal{P}(\mathcal{L}_J)$ satisfy

(1) $x \leq y$ and $x \Vdash p$ imply $y \Vdash p$,
(2) $x \leq y$ and $\phi \in \mathcal{E}_t(x)$ imply $\phi \in \mathcal{E}_t(y)$,
(3) $x \leq y$ implies $\mathcal{R}[y] \subseteq \mathcal{R}[x]$ where $\mathcal{R}[x] := \{z \in F \mid (x, z) \in \mathcal{R}\}$,

as well as

(i) $\mathcal{E}_t(x) \sqsupset \mathcal{E}_s(x) \subseteq \mathcal{E}_{[t \cdot s]}(x)$,
(ii) $\mathcal{E}_t(x) \cup \mathcal{E}_s(x) \subseteq \mathcal{E}_{[t+s]}(x)$,
(iii) $\mathcal{R}$ is reflexive,
(iv) $\mathcal{R}$ is transitive,
(v) $\mathcal{E}_t(x) \subseteq \mathcal{E}_t(y)$ for $y \in \mathcal{R}[x]$ (monotonicity),
(vi) $t : \mathcal{E}_t(x) \subseteq \mathcal{E}_{!t}(x)$ (introspectivity),

for all $p \in Var$, $\phi \in \mathcal{L}_J$, $t, s \in Jt$ and $x, y \in F$.

Over an intuitionistic Fitting model $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{R}, \mathcal{E}, \Vdash \rangle$, we introduce the following local satisfaction relation by recursion:

- $(\mathfrak{M}, x) \not\models \bot$;
- $(\mathfrak{M}, x) \models p$ if $x \Vdash p$;
- $(\mathfrak{M}, x) \models \phi \land \psi$ if $(\mathfrak{M}, x) \models \phi$ and $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \lor \psi$ if $(\mathfrak{M}, x) \models \phi$ or $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \to \psi$ if $\forall y \in F\, (x \leq y \Rightarrow (\mathfrak{M}, x) \not\models \phi$ or $(\mathfrak{M}, x) \models \psi)$;
- $(\mathfrak{M}, x) \models t : \phi$ if $\phi \in \mathcal{E}_t(x)$ and $\forall y \in \mathcal{R}[x]\, ((\mathfrak{M}, y) \models \phi)$.

Let C be a class of Kripke frames. Then, we write CKFLP for the class of all intuitionistic Fitting models over frames from C. We write $(\mathfrak{M}, x) \models \Gamma$ if $(\mathfrak{M}, x) \models \gamma$ for all $\gamma \in \Gamma$. Also, given an intuitionistic Fitting model $\mathfrak{M}$ over a Kripke frame $\mathfrak{F} = \langle F, \leq \rangle$, we write again $\mathcal{D}(\mathfrak{M}) = F$.

Again, the monotonicity property is immediate, i.e. for any $\phi \in \mathcal{L}_J$ and all $x, y \in F$:

$$x \leq y \text{ and } (\mathfrak{M}, x) \models \phi \text{ imply } (\mathfrak{M}, y) \models \phi.$$

Further, given some constant specification CS, we say that $\mathfrak{M}$ *respects* CS if for all $x \in F$ and all $c : \phi \in CS$, we have $\phi \in \mathcal{E}_c(x)$. Similar to before, given a class C of intuitionistic Fitting models, we write C_{CS} for the subclass of all models from C respecting CS.

Definition 5.10. Let C be a class of intuitionistic Fitting models. We write $\Gamma \models_{\mathsf{C}} \phi$ if for all $\mathfrak{M} \in \mathsf{C}$ and all $x \in \mathcal{D}(\mathfrak{M})$: $(\mathfrak{M}, x) \models \Gamma$ implies $(\mathfrak{M}, x) \models \phi$.

As before, soundness is immediate.

Lemma 5.11. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Let* $\mathsf{C} \in \mathsf{KFr}(\mathbf{L})$*. Then, for any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \textit{ implies } \Gamma \models_{\mathsf{CKFLP}_{CS}} \phi.$$

5.4. **Intuitionistic subset models.** The last semantics which we introduce, based on intuitionistic Kripke frames, extends the considerations of Lehmann and Studer from [20] about their subset models to these intermediate cases. This semantics seems to have not appeared in the literature before.

In that case, the relation $\Vdash$ of the model will already be defined on $F \times \mathcal{L}_J$ with suitable conditions and we consequently will not introduce an external $\models$ as before. The reason for this is that the subset models need $\Vdash$ to be defined on 'irregular' worlds where the usual semantical interpretations of the propositional and modal connectives are not respected (see also [20]).

Definition 5.12. Let $\mathfrak{F} = \langle F_0, \leq \rangle$ be a Kripke frame. An *intuitionistic subset model over* $\mathfrak{F}$ is a structure $\mathfrak{M} = \langle \mathfrak{F}, F, \mathcal{E}, \Vdash \rangle$ with $F \supseteq F_0$, $\mathcal{E} : Jt \to \mathcal{P}(F \times F)$ and $\Vdash \subseteq F \times \mathcal{L}_J$ and which satisfies

(1) $x \leq y$ and $x \Vdash p$ imply $y \Vdash p$,
(2) $x \leq y$ implies $\mathcal{E}_t[y] \subseteq \mathcal{E}_t[x]$ where $\mathcal{E}_t[x] := \{z \in F \mid (x, z) \in \mathcal{E}_t\}$,

as well as

(i) $x \not\Vdash \bot$,
(ii) $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$,
(iii) $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$,
(iv) $x \Vdash \phi \to \psi$ iff $\forall y \geq x\, (y \not\Vdash \phi$ or $y \Vdash \psi)$,
(v) $x \Vdash t : \phi$ iff $\forall y \in \mathcal{E}_t[x]\, (y \Vdash \phi)$,

and such that it satisfies

(a) $\mathcal{E}_{[t+s]}[x] \subseteq \mathcal{E}_t[x] \cap \mathcal{E}_s[x]$,
(b) $\mathcal{E}_{[t \cdot s]}[x] \subseteq \{y \in F \mid \forall \phi \in (\mathfrak{M})^x_{t,s} (y \Vdash \phi)\}$ where we define

$$(\mathfrak{M})^x_{t,s} := \{\phi \in \mathcal{L}_J \mid \exists \psi \in \mathcal{L}_J \forall y \in F(y \in \mathcal{E}_t[x] \Rightarrow y \Vdash \psi \to \phi \text{ and } y \in \mathcal{E}_s[x] \Rightarrow y \Vdash \psi)\},$$

(c) $x \in \mathcal{E}_t[x]$ (reflexivity);
(d) $\mathcal{E}_{!t}[x] \subseteq \{y \in F \mid \forall \phi \in \mathcal{L}_J (x \Vdash t : \phi \Rightarrow y \Vdash t : \phi)\}$ (introspectivity).

for all $p \in Var$, $\phi, \psi \in \mathcal{L}_J$, $t, s \in Jt$ and $x, y \in F_0$.

Given a class C of Kripke frames, we write CKSLP for the class of all intuitionistic subset models over frames from C. We write $\mathcal{D}_0(\mathfrak{M})$ for F_0 and $\mathcal{D}(\mathfrak{M})$ for F. Also, given $x \in \mathcal{D}(\mathfrak{M})$, we write $(\mathfrak{M}, x) \models \phi$ if $x \Vdash \phi$ and $(\mathfrak{M}, x) \models \Gamma$ if $(\mathfrak{M}, x) \models \gamma$ for all $\gamma \in \Gamma$, given $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We write $\mathfrak{M} \models \phi$ if for all $x \in \mathcal{D}_0(\mathfrak{M})$, we have $(\mathfrak{M}, x) \models \phi$ and similarly for sets Γ. Note the emphasis on $\mathcal{D}_0(\mathfrak{M})$, not $\mathcal{D}(\mathfrak{M})$.

Also here, intuitionistic subset models are monotone, i.e. for all $\phi \in \mathcal{L}_J$ and all $x, y \in F_0$:

$$x \leq y \text{ and } x \Vdash \phi \text{ imply } y \Vdash \phi.$$

Further, given a constant specification CS, we say that $\mathfrak{M}$ *respects* CS if $x \Vdash c : \phi$ for all $c : \phi \in CS$ and all $x \in F_0$. Given a class C of intuitionistic subset models, we write C_{CS} for the subclass of all models from C respecting the constant specification CS.

Definition 5.13. Let C be a class of intuitionistic subset models. We write $\Gamma \models_{\mathsf{C}} \phi$ if for all $\mathfrak{M} \in \mathsf{C}$ and all $x \in \mathcal{D}_0(\mathfrak{M})$: $(\mathfrak{M}, x) \models \Gamma$ implies $(\mathfrak{M}, x) \models \phi$.

Lemma 5.14. *Let* $\mathbf{L}$ *be an intermediate logic. Let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$ *and let* $\mathsf{C} \in \mathsf{KFr}(\mathbf{L})$*. Then, for any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \textit{ implies } \Gamma \models_{\mathsf{CKSLP}_{CS}} \phi.$$

6. Completeness for frame semantics

In the following, we again fix an $\mathbf{L}$ and a constant specification CS for $\mathbf{LP}_0(\mathbf{L})$. Then, the approach to the uniform completeness results is very similar to the algebraic case: given $\Gamma \not\vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi$, we obtain

$$[\Gamma]^\star \cup [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star \not\vdash_{\mathbf{L}^\star} \phi^\star$$

via Lemma 4.1 as before and assuming a class of Kripke frames C which is globally complete w.r.t. $\mathbf{L}$, we then obtain a model $\mathfrak{N} \in \mathrm{Mod}(\mathfrak{F}; \mathcal{L}_0^\star)$ with $\mathfrak{F} \in \mathsf{C}$ such that $\mathfrak{N} \models [\Gamma]^\star$ and $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ but $\mathfrak{N} \not\models \phi^\star$.

This $\mathfrak{N}$ is then used to define the associated intuitionistic Mkrtychev, Fitting or subset model over $\mathfrak{F}$ such that evaluation ϕ is true at the point $x \in \mathfrak{F}$ in the model iff $(\mathfrak{N}, x) \models \phi^\star$. The property $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ guarantees well-definedness in a similar way as before (which is also why we rely on the *global* completeness statement introduced earlier as $(\mathfrak{N}, x) \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ is needed *for all* worlds x).

As before, this construction does not rely on any feature of the class C of frames other than strong local and global completeness w.r.t. $\mathbf{L}$ which implies a similar degree of uniformity as before.

6.1. Completeness w.r.t. intuitionistic Mkrtychev models.

Definition 6.1. Let $\mathfrak{F} = \langle F, \leq\rangle$ be a Kripke frame and let $\mathfrak{N} = \langle \mathfrak{F}, \Vdash^*\rangle \in \mathsf{Mod}(\mathfrak{F}; \mathcal{L}_0^\star)$. We define the *canonical intuitionistic Mkrtychev model over* $\mathfrak{N}$ as the structure $\mathfrak{M}^{c,M}_{\mathfrak{N}} = \langle \mathfrak{F}, \mathcal{E}^c, \Vdash^c\rangle$ by setting:

(1) $x \Vdash^c p$ iff $x \Vdash^* p$;
(2) $\mathcal{E}^c_t(x) := \{\phi \in \mathcal{L}_J \mid x \Vdash^* \phi_t\}$.

Lemma 6.2. *Let $\mathfrak{F} = \langle F, \leq\rangle$ be a Kripke frame and let $\mathfrak{N} = \langle \mathfrak{F}, \Vdash^*\rangle \in \mathsf{Mod}(\mathfrak{F}; \mathcal{L}_0^\star)$ such that additionally $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$. Then, for all $\phi \in \mathcal{L}_J$ and all $x \in F$:*

$$(\mathfrak{M}^{c,M}_{\mathfrak{N}}, x) \models \phi \text{ iff } (\mathfrak{N}, x) \models \phi^\star.$$

Further, $\mathfrak{M}^{c,M}_{\mathfrak{N}}$ is a well-defined intuitionistic Mkrtychev model respecting CS.

Proof. The equivalence of $\models$ for $\mathfrak{N}$ and $\mathfrak{M}^{c,M}_{\mathfrak{N}}$ can be shown by a straightforward induction on ϕ.

The monotonicity properties (1) and (2) immediately follow from monotonicity of $\mathfrak{N}$.

For property (i), let $\phi \in \mathcal{E}^c_t(x) \sqsupset \mathcal{E}^c_s(x)$, i.e.

$$\exists \psi \in \mathcal{L}_J \left(\psi \to \phi \in \mathcal{E}^c_t(x) \text{ and } \psi \in \mathcal{E}^c_s(x)\right).$$

Then, by definition, we have

$$x \Vdash^* (\psi \to \phi)_t \text{ and } x \Vdash^* \psi_s$$

and thus, as $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$, we get

$$x \Vdash^* \phi_{[t\cdot s]},$$

that is $\phi \in \mathcal{E}^c_{t\cdot s}(x)$.

For property (ii), note that $x \Vdash^* \phi_t$ implies $x \Vdash^* \phi_{[t+s]}$ again by $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$, hence $\phi \in \mathcal{E}^c_t(x)$ implies $\phi \in \mathcal{E}^c_{t+s}(x)$ and similarly for $\phi \in \mathcal{E}^c_s(x)$.

For (iii), note that $\phi \in \mathcal{E}^c_t(x)$ implies $x \Vdash^* \phi_t$ by definition, i.e. $(\mathfrak{N}, x) \models \phi^\star$ by axiom (F) and $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$. The first claim of the lemma gives $(\mathfrak{M}^{c,M}_{\mathfrak{N}}, x) \models \phi$

For (iv), let $\phi \in \mathcal{E}^c_t(x)$. Then $x \Vdash^* \phi_t$ by axiom (I), we have $x \Vdash^* (t : \phi)_{!t}$ since $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$, i.e. $t : \phi \in \mathcal{E}^c_{!t}(x)$. ☐

Following the general outline from above immediately yields the following completeness result.

Theorem 6.3. *Let $\mathbf{L}$ be an intermediate logic and let CS be a constant specification for $\mathbf{LP}_0(\mathbf{L})$. Let $\mathsf{C} \in \mathsf{KFr}(\mathbf{L}) \cap \mathsf{KFr}^g(\mathbf{L})$. Then for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, we have:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \text{ iff } \Gamma \models_{\mathsf{CKMLP}_{CS}} \phi.$$

6.2. Completeness w.r.t. intuitionistic Fitting models.

Definition 6.4. Let $\mathfrak{F} = \langle F, \leq\rangle$ be a Kripke frame and let $\mathfrak{N} = \langle \mathfrak{F}, \Vdash^*\rangle \in \mathsf{Mod}(\mathfrak{F}; \mathcal{L}_0^\star)$. We define the *canonical intuitionistic Fitting model over* $\mathfrak{N}$ as the structure $\mathfrak{M}^{c,F}_{\mathfrak{N}} = \langle \mathfrak{F}, \mathcal{R}^c, \mathcal{E}^c, \Vdash^c\rangle$ by setting:

(1) $x \Vdash^c p$ iff $x \Vdash^* p$;
(2) $\mathcal{E}^c_t(x) := \{\phi \in \mathcal{L}_J \mid x \Vdash^* \phi_t\}$;
(3) $(x, y) \in \mathcal{R}^c$ iff $\forall t \in Jt \forall \phi \in \mathcal{L}_J\, (x \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, y) \models \phi^\star)$.

Lemma 6.5. *Let $\mathfrak{F} = \langle F, \leq\rangle$ be a Kripke frame, let $\mathfrak{N} = \langle \mathfrak{F}, \Vdash^*\rangle \in \mathsf{Mod}(\mathfrak{F}; \mathcal{L}_0^\star)$ such that $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$. Then, for any $\phi \in \mathcal{L}_J$ and all $x \in F$:*

$$(\mathfrak{M}^{c,F}_{\mathfrak{N}}, x) \models \phi \text{ iff } (\mathfrak{N}, x) \models \phi^\star.$$

Further, $\mathfrak{M}^{c,F}_{\mathfrak{N}}$ is a well-defined intuitionistic Fitting model respecting CS.

Proof. The first claim is again proved by induction on the structure of the formula. We only consider the modal case. Suppose the claim holds for all $x \in F$ and some $\phi \in \mathcal{L}_J$. At first, suppose $(\mathfrak{N}, x) \models \phi_t$, i.e. $x \Vdash^* \phi_t$. Then, naturally $\phi \in \mathcal{E}^c_t(x)$ by definition. Further, let $y \in \mathcal{R}^c[x]$. Then, as $x \Vdash^* \phi_t$, we have $(\mathfrak{N}, y) \models \phi^\star$ by definition and thus $(\mathfrak{M}^{c,F}_{\mathfrak{N}}, y) \models \phi$ by induction hypothesis. Hence, we get

$$\phi \in \mathcal{E}^c_t(x) \text{ and } \forall y \in \mathcal{R}^c[x] \left((\mathfrak{M}^{c,F}_{\mathfrak{N}}, y) \models \phi \right)$$

and consequently $(\mathfrak{M}^{c,F}_{\mathfrak{N}}, x) \models t : \phi$.

Conversely, suppose $(\mathfrak{N}, x) \not\models \phi_t$, that is $x \not\Vdash^* \phi_t$. Then $\phi \notin \mathcal{E}^c_t(x)$ by definition and thus

$$(\mathfrak{M}^{c,F}_{\mathfrak{N}}, x) \not\models t : \phi$$

immediately by definition.

For the monotonicity properties (1) - (3), we only sketch (3): let $z \in \mathcal{R}[y]$, that is we have

$$\forall t \in Jt \forall \phi \in \mathcal{L}_J \ (y \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, z) \models \phi^\star).$$

Then, for any $t \in Jt$ and any $\phi \in \mathcal{L}_J$ we have, if $x \Vdash^* \phi_t$ that $y \Vdash^* \phi_t$ by monotonicity of $\mathfrak{N}$ and thus $(\mathfrak{N}, z) \models \phi^\star$. Hence, $z \in \mathcal{R}^c[x]$ and so $\mathcal{R}^c[y] \subseteq \mathcal{R}^c[x]$.

Items (i) and (ii) can be shown as in the case of Mkrtychev models.

As (F) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, we have

$$\forall t \in Jt \forall \phi \in \mathcal{L}_J \ (x \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, x) \models \phi^\star)$$

as $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ and this is exactly $(x, x) \in \mathcal{R}^c$.

As (I) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, one can easily see that

$$t : \mathcal{E}^c_t(x) \subseteq \mathcal{E}^c_{!t}(x)$$

as in the Mkrtychev case. For the transitivity of $\mathcal{R}^c$, let $(x, y), (y, z) \in \mathcal{R}^c$, that is, we have

$$\forall t \in Jt \forall \phi \in \mathcal{L}_J \ (x \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, y) \models \phi^\star)$$

as well as

$$\forall t \in Jt \forall \phi \in \mathcal{L}_J \ (y \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, z) \models \phi^\star).$$

By $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$, we have

$$w \Vdash^* \phi_t \Rightarrow w \Vdash^* (t : \phi)_{!t}$$

for any $w \in F$. Thus, in particular, we have

$$x \Vdash^* \phi_t \Rightarrow x \Vdash^* (t : \phi)_{!t} \Rightarrow y \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, z) \models \phi^\star$$

via the first claim. By definition, this yields $(x, z) \in \mathcal{R}^c$.

For the monotonicity, let $y \in \mathcal{R}^c[x]$ and let $\phi \in \mathcal{E}_t(x)$. The former gives

$$\forall t \in Jt \forall \phi \in \mathcal{L}_J \ (x \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, y) \models \phi^\star)$$

and the latter gives $x \Vdash^* \phi_t$. As $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$, we have especially $x \Vdash^* (t : \phi)_{!t}$. By the above, this gives us $(\mathfrak{N}, y) \models \phi_t$, that is $y \Vdash^* \phi_t$ and thus $\phi \in \mathcal{E}^c_t(y)$. Hence, $\mathfrak{M}^{c,F}_{\mathfrak{N}}$ is monotone and it follows that $\mathfrak{M}^{c,F}_{\mathfrak{N}}$ is introspective. □

Theorem 6.6. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$. *Let* $\mathsf{C} \in \mathsf{KFr}(\mathbf{L}) \cap \mathsf{KFr}^g(\mathbf{L})$. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, *we have:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \textit{ iff } \Gamma \models_{\mathsf{CKFLP}_{CS}} \phi.$$

6.3. **Completeness w.r.t. intuitionistic subset models.** The construction for the canonical intuitionistic subset model is a little bit more subtle. For a Kripke frame $\langle F_0, \leq \rangle$, we add irregular worlds x_t for $t \in Jt$ and $x \in F_0$ which take a similar role as the previously used construction v_t for evaluations v.

Definition 6.7. Let $\mathfrak{F} = \langle F_0, \leq \rangle$ be a Kripke frame and let $\mathfrak{N} = \langle \mathfrak{F}, \Vdash^* \rangle \in \mathsf{Mod}(\mathfrak{F}; \mathcal{L}^\star_0)$. We define the *canonical intuitionistic subset model over* $\mathfrak{N}$ as the structure $\mathfrak{M}^{c,S}_{\mathfrak{N}} = \langle \mathfrak{F}, F^c, \mathcal{E}^c, \Vdash^c \rangle$ by setting:

1. $F^c = F_0 \cup \bigcup_{x \in F_0} \{x_t \mid t \in Jt\}$;
2. $(x, y) \in \mathcal{E}^c_t$ iff $\forall \phi \in \mathcal{L}_J \ (x \Vdash^* \phi_t \Rightarrow y \Vdash^c \phi)$ for all $x, y \in F^c$;
3. for $x \in F_0$ and $t \in Jt$:
   (a) $x \Vdash^c \phi$ iff $(\mathfrak{N}, x) \models \phi^\star$;
   (b) $x_t \Vdash^c \phi$ iff $(\mathfrak{N}, x) \models \phi_t$.

Lemma 6.8. *Let $\mathfrak{F} = \langle F_0, \leq \rangle$ be a Kripke frame, let $\mathfrak{N} \in \mathsf{Mod}(\mathfrak{F}; \mathcal{L}_0^\star)$ such that $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$. Then, for any $\phi \in \mathcal{L}_J$ and any $x \in F_0$:*

$$(\mathfrak{M}^{c,S}_{\mathfrak{N}}, x) \models \phi \text{ iff } (\mathfrak{N}, x) \models \phi^\star.$$

Further, $\mathfrak{M}^{c,S}_{\mathfrak{N}}$ is a well-defined intuitionistic subset model.

Proof. The first claim is clear by definition as we have $(\mathfrak{M}^{c,S}_{\mathfrak{N}}, x) \models \phi$ iff $x \Vdash^c \phi$ iff $(\mathfrak{N}, x) \models \phi^\star$, given a $x \in F_0$.

Let $x \in F_0$. The properties (i) - (iv) are immediate by using the respective properties of $\Vdash^*$ and the fact that $\star$ commutes with $\bot, \wedge, \vee$ and since $\leq$ is only an order on F_0.

For (v), we have for one by definition that

$$\forall y \in \mathcal{E}^c_t[x] \forall \phi \in \mathcal{L}_J \left((\mathfrak{N}, x) \models \phi_t \Rightarrow (\mathfrak{N}, y) \models \phi^\star\right),$$

that is we have

$$\begin{aligned} x \Vdash^c t : \phi &\Rightarrow x \Vdash^* \phi_t \\ &\Rightarrow \forall y \in \mathcal{E}^c_t[x] \left((\mathfrak{N}, y) \models \phi^\star\right) \\ &\Leftrightarrow \forall y \in \mathcal{E}^c_t[x] \left(y \Vdash^c \phi\right). \end{aligned}$$

For another, we have $x_t \in \mathcal{E}_t[x]$ as we know $x_t \Vdash^c \phi$ iff $x \Vdash^* \phi_t$ by definition. Thus, if $x \nVdash^c t : \phi$, then $x \nVdash^* \phi_t$ and $x_t \nVdash^c \phi$. Therefore

$$x \nVdash^c t : \phi \Rightarrow \exists y \in \mathcal{E}_t[x] \left(y \nVdash^c \phi\right).$$

Concluding, we have $x \Vdash^c t : \phi$ iff $\forall y \in \mathcal{E}^c_t[x] \left(y \Vdash^c \phi\right)$.

Monotonicity in form of properties (1) and (2) follows from monotonicity of $\mathfrak{N}$ and the previous claims.

For (a), let $y \in \mathcal{E}^c_{[t+s]}[x]$, that is we have

$$\forall \phi \in \mathcal{L}_J (x \Vdash^* \phi_{[t+s]} \Rightarrow (\mathfrak{N}, y) \models \phi^\star)$$

by definition. As $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ we have $x \Vdash^* \phi_t$ implies $x \Vdash^* \phi_{[t+s]}$ and $x \Vdash^* \phi_s$ implies $x \Vdash^* \phi_{[t+s]}$. Therefore, we obtain

$$\forall \phi \in \mathcal{L}_J (x \Vdash^* \phi_t \Rightarrow x \Vdash^* \phi_{[t+s]} \Rightarrow (\mathfrak{N}, y) \models \phi^\star)$$

and

$$\forall \phi \in \mathcal{L}_J (x \Vdash^* \phi_s \Rightarrow x \Vdash^* \phi_{[t+s]} \Rightarrow (\mathfrak{N}, y) \models \phi^\star)$$

which is $y \in \mathcal{E}^c_t[x] \cap \mathcal{E}^c_s[x]$.

For (b), let $y \in \mathcal{E}^c_{[t \cdot s]}[x]$, that is

$$(\dagger) \qquad \forall \phi \in \mathcal{L}_J (x \Vdash^* \phi_{[t \cdot s]} \Rightarrow (\mathfrak{N}, y) \models \phi^\star).$$

Let $\phi \in (\mathfrak{M}^{c,S}_{\mathfrak{N}})^x_{t,s}$, that is there is a $\psi \in \mathcal{L}_J$ such that

$$\forall z \in F^c (z \in \mathcal{E}^c_t[x] \Rightarrow z \Vdash^c \psi \to \phi \text{ and } z \in \mathcal{E}^c_s[x] \Rightarrow z \Vdash^c \psi).$$

By property (v), we have that $x \Vdash^c t : (\psi \to \phi)$ and $x \Vdash^c s : \psi$, i.e. by definition as $x \in F_0$:

$$(\mathfrak{N}, x) \models (\psi \to \phi)_t \text{ and } (\mathfrak{N}, x) \models \psi_s$$

and by $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ and axiom (J), we get

$$(\mathfrak{N}, x) \models \phi_{[t \cdot s]}.$$

Thus, by $(\dagger)$, we have $(\mathfrak{N}, y) \models \phi^\star$ and by definition this gives $y \Vdash^c \phi$.

As (F) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$, we have

$$x \Vdash^* \phi_t \Rightarrow (\mathfrak{N}, x) \models \phi^\star$$

for all $x \in F_0$ and all $\phi \in \mathcal{L}_J$, $t \in Jt$ as $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$ and thus, by definition we have $x \in \mathcal{E}^c_t[x]$ for all $t \in Jt$.

As (I) is an axiom scheme of $\mathbf{LP}_{CS}(\mathbf{L})$. Let $y \in \mathcal{E}_{!t}[x]$, that is

$$(\ddagger) \qquad \forall \phi \in \mathcal{L}_J (x \Vdash^* \phi_{!t} \Rightarrow (\mathfrak{N}, y) \models \phi^\star).$$

Let $\phi \in \mathcal{L}_J$ and assume $x \Vdash^c t : \phi$, that is $x \Vdash^* \phi_t$. Then, as $\mathfrak{N} \models [Th_{\mathbf{LP}_{CS}(\mathbf{L})}]^\star$, we have $x \Vdash^* (t : \phi)_{!t}$. By $(\ddagger)$ we get $(\mathfrak{N}, y) \models (t : \phi)^\star$, that is $(\mathfrak{N}, y) \models \phi_t$ and thus by definition $y \Vdash^c t : \phi$. □

This then also yields the following completeness theorem.

Theorem 6.9. *Let* $\mathbf{L}$ *be an intermediate logic and let* CS *be a constant specification for* $\mathbf{LP}_0(\mathbf{L})$*. Let* $\mathsf{C} \in \mathsf{KFr}(\mathbf{L}) \cap \mathsf{KFr}^g(\mathbf{L})$*. Then, for any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*, we have:*

$$\Gamma \vdash_{\mathbf{LP}_{CS}(\mathbf{L})} \phi \text{ iff } \Gamma \models_{\mathsf{CKSLP}_{CS}} \phi.$$

7. Intermediate Modal Logics

One of the main themes in the theory of justification logics is of course their strong correspondence with non-explicit modal logics via realization. In our case, we will see that the $\mathbf{LP}(\mathbf{L})$ are the natural explicit correspondents to the $\mathbf{L}$-intermediate version of the modal logic $\mathbf{S4}$, generalizing the results from the classical and intuitionistic case. For that, we first give an overview of the (semantic) theory of intermediate modal logics (in some way following [25, 32]).

To define intermediate modal logics, we consider a usual modal language with a single modality $\Box$ given by

$$\mathcal{L}_\Box : \phi ::= \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid \Box\phi$$

where again $p \in Var$. Propositional substitutions naturally generalize to $\sigma : Var \to \mathcal{L}_\Box$ and we still write $\sigma(\phi)$ for the image of ϕ under the natural extension of σ to $\mathcal{L}_\Box$ by commuting with $\wedge, \vee, \to, \bot$ and $\Box$.

Definition 7.1. A (normal) *intermediate modal logic* is a set $\mathbf{IML} \subsetneq \mathcal{L}_\Box$ such that

1. $\mathbf{IPC} \cup (K) \subseteq \mathbf{IML}$ where (K) is given by
$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi),$$
2. $\mathbf{IML}$ is closed under substitution in $\mathcal{L}_\Box$,
3. $\mathbf{IML}$ is closed under modus ponens,
4. $\mathbf{IML}$ is closed under *necessitation*, that is $\phi \in \mathbf{IML}$ implies $\Box\phi \in \mathbf{IML}$.

We denote the smallest such logic by $\mathbf{K}(\mathbf{IPC})$. To give more direct definitions of axiomatic extensions, we introduce the following notation. Given sets $\Gamma, \Delta \subseteq \mathcal{L}_\Box$, we write $\Gamma \oplus \Delta$ for the *normal closure of* $\Gamma \cup \Delta$, that is for the smallest set $\Phi \subseteq \mathcal{L}_\Box$ with $\Gamma \cup \Delta \subseteq \Phi$ and which is closed under modus ponens, substitution in $\mathcal{L}_\Box$ and necessitation.

In particular, in the following, we consider the two axiom schemes

(T) $\Box\phi \to \phi$,
(4) $\Box\phi \to \Box\Box\phi$,

and, given an intermediate logic $\mathbf{L}$, we write

$$\mathbf{S4}(\mathbf{L}) := \mathbf{K}(\mathbf{IPC}) \oplus \mathbf{L} \oplus (T) \oplus (4)$$

As before with the intermediate justification logics, given a set $\Gamma \subseteq \mathcal{L}_\Box$ and an intermediate modal logic $\mathbf{IML}$, we write

$$\Gamma \vdash_{\mathbf{IML}} \phi \text{ iff } \exists \gamma_1, \dots, \gamma_n \left(\bigwedge_{i=1}^n \gamma_i \to \phi \in \mathbf{IML} \right).$$

As we focus on the particular family $\mathbf{LP}(\mathbf{L})$ of intermediate justification logics, the main emphasis on the modal side will be on the logics $\mathbf{S4}(\mathbf{L})$ introduced above. However, all the following results extend to natural intermediate counterparts of the modal logics $\mathbf{K}$, $\mathbf{T}$, $\mathbf{4}$ over some intermediate logic $\mathbf{L}$ and corresponding fragments of $\mathbf{LP}(\mathbf{L})$, i.e. $\mathbf{J}(\mathbf{L})$, $\mathbf{JT}(\mathbf{L})$ and $\mathbf{J4}(\mathbf{L})$, respectively.

7.1. Semantics and Completeness.
We will need some semantical notions regarding intermediate modal logics for the to-follow model theoretical considerations regarding realizations. For this, we introduce so called intuitionistic modal Kripke models which go back to Ono's work [25].

Definition 7.2. An *intuitionistic modal Kripke frame* is a structure $\mathfrak{F} = \langle F, \leq, \mathcal{R}\rangle$ where $\langle F, \leq\rangle$ is a partial order and $\mathcal{R} \subseteq F \times F$ with

$$x \leq y \Rightarrow \mathcal{R}[y] \subseteq \mathcal{R}[x].$$

An *intuitionistic modal Kripke model* over $\mathfrak{F}$ is a structure $\mathfrak{M} = \langle \mathfrak{F}, \Vdash\rangle$ where $\Vdash \subseteq F \times Var$ such that

$$(x \Vdash p \text{ and } x \leq y) \Rightarrow y \Vdash p.$$

We write $\mathcal{D}(\mathfrak{M}) := \mathcal{D}(\mathfrak{F}) := F$. Further, given a class C of intuitionistic modal Kripke frames, we write $\mathsf{Mod}(\mathsf{C})$ for the class of all models over these frames.

Let $\mathfrak{M} = \langle F, \leq, \mathcal{R}, \Vdash\rangle$ be an intuitionistic modal Kripke model and let $x \in \mathcal{D}(\mathfrak{M})$. We define the relation $\models$ recursively as follows:

- $(\mathfrak{M}, x) \not\models \bot$;
- $(\mathfrak{M}, x) \models p$ iff $x \Vdash p$ for $p \in Var$;

- $(\mathfrak{M}, x) \models \phi \wedge \psi$ iff $(\mathfrak{M}, x) \models \phi$ and $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \vee \psi$ iff $(\mathfrak{M}, x) \models \phi$ or $(\mathfrak{M}, x) \models \psi$;
- $(\mathfrak{M}, x) \models \phi \rightarrow \psi$ iff $\forall y \geq x\,((\mathfrak{M}, y) \models \phi$ implies $(\mathfrak{M}, y) \models \psi)$;
- $(\mathfrak{M}, x) \models \Box\phi$ iff $\forall y \in \mathcal{R}[x]\,((\mathfrak{M}, y) \models \phi)$.

We write $(\mathfrak{M}, x) \models \Gamma$ if $(\mathfrak{M}, x) \models \gamma$ for all $\gamma \in \Gamma$, given $\Gamma \subseteq \mathcal{L}_\Box$.

Definition 7.3. Let C be a class of intuitionistic modal Kripke models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$. We write $\Gamma \models_{\mathsf{C}} \phi$ if

$$\forall \mathfrak{M} \in \mathsf{C} \forall x \in \mathcal{D}(\mathfrak{M}) \Big((\mathfrak{M}, x) \models \Gamma \Rightarrow (\mathfrak{M}, x) \models \phi \Big).$$

If C is a class of intuitionistic modal Kripke frames, we write $\Gamma \models_{\mathsf{C}} \phi$ if $\Gamma \models_{\mathsf{Mod(C)}} \phi$.

Definition 7.4. Let **IML** be an intermediate modal logic. **IML** is *(strongly) Kripke complete w.r.t. a class* C of intuitionistic modal frames, if

$$\Gamma \vdash_{\mathbf{IML}} \phi \text{ iff } \Gamma \models_{\mathsf{C}} \phi$$

for all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$.

We write $\mathsf{C} \in \mathsf{MFr}(\mathbf{IML})$ in the above case. As there are Kripke incomplete intermediate logics (recall [30]), there are Kripke incomplete intermediate modal logics. So again, the above notation is meant to be read as if it includes an existence statement.

8. Substitutions and Realization

As touched upon in the introduction, our approach for establishing a realization theorem for the intermediate justification logics and the previously introduced intermediate modal logics is based on Fitting's work [9] about a semantic, but non-constructive, proof of the classical realization theorems. We adapt this proof using the previously introduced intuitionistic Fitting models.

The central points of this approach to the realization theorem are, for one, using a specific canonical model for the (intermediate) justification logic and translating this to an appropriate model for the (intermediate) modal logic to obtain the existence of so-called quasi-realizations (which will be precisely defined later on). The second central point is a constructive extraction of normal realizations from quasi-realizations using a recursion on the structure of the modal formula.

For all these considerations, the following subsections introduce some technical tools.

8.1. **Annotated Modal Formulas.** To keep track over different $\Box$-symbols in modal formulas, which are potentially realized by different justification terms, we consider an annotated modal language given by

$$\mathcal{L}'_\Box : \phi' ::= \bot \mid p \mid (\phi' \wedge \phi') \mid (\phi' \vee \phi') \mid (\phi' \rightarrow \phi') \mid \Box_n \phi'$$

with $n \in \mathbb{N}$ and $p \in Var$. There is a natural projection from $\mathcal{L}'_\Box$ to $\mathcal{L}_\Box$ by dropping all $\Box$-annotations and we denote it by $(\cdot)^\bullet$. Followingly, we call an annotated formula $\chi \in \mathcal{L}'_\Box$ an annotation of a non-annotated formula ϕ if $\chi^\bullet = \phi$ and then also often write ϕ' for χ.

We call an annotation ϕ' uniquely annotated if no $\Box$-index occurs more than once.

8.2. **Substitutions and Realizations.** We define the operation $[\![\cdot]\!]$ (following Fitting's work [9]), collecting all possible realizations of an annotated formula, given extra information on polarity.

More precisely, we define $[\![\cdot]\!] : \{T, F\} \times \mathcal{L}'_\Box \rightarrow \mathcal{P}(\{T, F\} \times \mathcal{L}_J)$ by recursion on $\mathcal{L}'_\Box$:

(1) For $\phi' \in Var \cup \{\bot\}$:
$[\![T, \phi']\!] := \{(T, \phi')\}$; $[\![F, \phi']\!] = \{(F, \phi')\}$;

(2) For $\circ \in \{\wedge, \vee\}$:
$[\![T, \phi' \circ \psi']\!] := \{(T, \alpha \circ \beta) \mid (T, \alpha) \in [\![T, \phi']\!], (T, \beta) \in [\![T, \psi']\!]\}$;
$[\![F, \phi' \circ \psi']\!] := \{(F, \alpha \circ \beta) \mid (F, \alpha) \in [\![F, \phi']\!], (F, \beta) \in [\![F, \psi']\!]\}$;

(3) $[\![T, \phi' \rightarrow \psi']\!] := \{(T, \alpha \rightarrow \beta) \mid (F, \alpha) \in [\![F, \phi']\!], (T, \beta) \in [\![T, \psi']\!]\}$;
$[\![F, \phi' \rightarrow \psi']\!] := \{(F, \alpha \rightarrow \beta) \mid (T, \alpha) \in [\![T, \phi']\!], (F, \beta) \in [\![F, \psi']\!]\}$;

(4) $[\![T, \Box_n \phi']\!] := \{(T, x_n : \alpha) \mid (T, \alpha) \in [\![T, \phi']\!]\}$;
$[\![F, \Box_n \phi']\!] := \{(F, t : \alpha) \mid (F, \alpha) \in [\![F, \phi']\!], t \in Jt\}$.

Recalling a comment from [9], we remark that the symbols T, F stem from the proof theoretic context of using tableau theorem proving. Here however, they are used just as syntactic bookkeeping of polarities.

Definition 8.1. A *realization of a formula* $\phi \in \mathcal{L}_\Box$ is any ψ with $(F, \psi) \in [\![F, \phi']\!]$ where ϕ' is an unique annotation of ϕ.

Definition 8.2. A *justification substitution* is a function $\sigma : V \to Jt$. This function naturally extends to a function $\sigma : Jt \to Jt$ by

(1) $\sigma([t * s]) := \sigma(t) * \sigma(s)$ for $* \in \{\cdot, +\}$,
(2) $\sigma(!t) :=!\sigma(t)$.

Further, σ also extends to a function $\sigma : \mathcal{L}_J \to \mathcal{L}_J$ by commuting with all connectives $\wedge, \vee, \to, \bot$ and setting

$$\sigma(t : \phi) := \sigma(t) : \sigma(\phi).$$

Note, that we write σ throughout for all extensions.

Given a formula ϕ or term t, we also write $\phi\sigma$ or $t\sigma$ for the image of it under a justification substitution σ.

Lemma 8.3. *Let* $\mathbf{L}$ *be an intermediate logic. If* $\mathbf{LP}(\mathbf{L}) \vdash \phi$*, then* $\mathbf{LP}(\mathbf{L}) \vdash \phi\sigma$ *for any justification substitution* σ*.*

The proof is similar to the classical case (see e.g. [19]). For the upcoming investigations, we will need some further vocabulary regarding justification substitutions. At first, given a justification formula $\phi \in \mathcal{L}_J$ or justification term $t \in Jt$, we write $\mathrm{jvar}(\phi)$ or $\mathrm{jvar}(t)$ for the sets of all justification variables $x \in V$ occurring in ϕ or t, respectively.

Definition 8.4 (Fitting [9])**.** Let σ be a justification substitution and ϕ' be a uniquely annotated formula. We write $\mathrm{dom}(\sigma) = \{x \in V \mid x\sigma \neq x\}$. Further, we say:

(1) σ meets the *no-new-variables* condition if $\mathrm{jvar}(x\sigma) \subseteq \{x\}$ for all $x \in V$;
(2) σ *lives on* ϕ' if $x_k \in \mathrm{dom}(\sigma)$ implies that $\Box_k$ occurs in ϕ';
(3) σ *lives away from* ϕ' if $x_k \in \mathrm{dom}(\sigma)$ implies that $\Box_k$ does not occur in ϕ'.

There is a natural way of combining justification substitutions by iteratively applying them. For this, given two justification substitutions σ, σ', we write $\sigma\sigma'$ for the substitution defined by

$$x \mapsto \sigma'(\sigma(x))$$

for all $x \in V$.

Lemma 8.5 (Fitting [9])**.** *Let* $\phi' \in \mathcal{L}'_\Box$ *be uniquely annotated,* σ_0 *a justification substitution that lives on* ϕ' *and* σ_1 *be a justification substitution that lives away from* ϕ'*. Then:*

(1) $(T, \alpha) \in [\![T, \phi']\!]$ *implies* $(T, \alpha\sigma_1) \in [\![T, \phi']\!]$*;*
$(F, \alpha) \in [\![F, \phi']\!]$ *implies* $(F, \alpha\sigma_1) \in [\![F, \phi']\!]$*.*
(2) If σ_0, σ_1 *meet the no-new-variable condition, then* $\sigma_0\sigma_1 = \sigma_1\sigma_0$*.*

A proof can also be found in [9].

9. Unified Quasi-Realizations for Intermediate Modal Logics

Following Fitting in [9], we define the mapping $\langle\!\langle \cdot \rangle\!\rangle : \{T, F\} \times \mathcal{L}'_\Box \to \mathcal{P}(\{T, F\} \times \mathcal{L}_J)$ by recursion on $\mathcal{L}'_\Box$:

(1) For $\phi' \in Var \cup \{\bot\}$:
$\langle\!\langle T, \phi' \rangle\!\rangle := \{(T, \phi')\}$; $\langle\!\langle F, \phi' \rangle\!\rangle = \{(F, \phi')\}$;

(2) For $\circ \in \{\wedge, \vee\}$:
$\langle\!\langle T, \phi' \circ \psi' \rangle\!\rangle := \{(T, \alpha \circ \beta) \mid (T, \alpha) \in \langle\!\langle T, \phi' \rangle\!\rangle, (T, \beta) \in \langle\!\langle T, \psi' \rangle\!\rangle\}$;
$\langle\!\langle F, \phi' \circ \psi' \rangle\!\rangle := \{(F, \alpha \circ \beta) \mid (F, \alpha) \in \langle\!\langle F, \phi' \rangle\!\rangle, (F, \beta) \in \langle\!\langle F, \psi' \rangle\!\rangle\}$;

(3) $\langle\!\langle T, \phi' \to \psi' \rangle\!\rangle := \{(T, \alpha \to \beta) \mid (F, \alpha) \in \langle\!\langle F, \phi' \rangle\!\rangle, (T, \beta) \in \langle\!\langle T, \psi' \rangle\!\rangle\}$;
$\langle\!\langle F, \phi' \to \psi' \rangle\!\rangle := \{(F, \bigwedge_{i=1}^k \alpha_i \to \bigvee_{j=1}^l \beta_j) \mid (T, \alpha_i) \in \langle\!\langle T, \phi' \rangle\!\rangle, (F, \beta_j) \in \langle\!\langle F, \psi' \rangle\!\rangle, i \leq k, j \leq l\}$;

(4) $\langle\!\langle T, \Box_n \phi' \rangle\!\rangle := \{(T, x_n : \alpha) \mid (T, \alpha) \in \langle\!\langle T, \phi' \rangle\!\rangle\}$;
$\langle\!\langle F, \Box_n \phi' \rangle\!\rangle := \{(F, t : (\alpha_1 \vee \cdots \vee \alpha_m)) \mid (F, \alpha_i) \in \langle\!\langle F, \phi' \rangle\!\rangle (i \leq m), t \in Jt\}$;

Definition 9.1. Given an uniquely annotated modal formula ϕ', a *quasi-realization for* ϕ' is a formula $\alpha_1 \vee \cdots \vee \alpha_n$ with $(F, \alpha_i) \in \langle\!\langle F, \phi' \rangle\!\rangle$ $(i \leq n)$. A quasi-realization for a modal formula ϕ is any quasi-realization for any unique annotation ϕ' of ϕ.

9.1. **Canonical models for intermediate justification logics, revisited.** The completeness proofs for intermediate justification logics (w.r.t. intuitionistic Fitting models) provided in Section 6 of this paper are relying on a kind of canonical model construction, relative, however, to a given intuitionistic frame (and a corresponding propositional model) to achieve the respective uniformity w.r.t. the class of frames.

Achieving this uniformity with the "usual" canonical model construction based on a frame over partially ordered maximal consistent sets or tableaux (such as the one defined later) is *sometimes* even impossible, as one can not always control the properties of the partial order such that it belongs to the desired class of frames.

For instance, the canonical model which we are about to present is, if constructed over a classical justification logic, not a single world model, but one with isolated single worlds w.r.t. the partial order. The previous completeness theorems, however, provide completeness for classical justification logic w.r.t. single world models based on the corresponding completeness theorem for classical propositional logic and single world frames.

This usual (or standard) canonical model construction is, however, the main tool of the present section as we need to have precise control over the frame of the (canonical) model in question. So, this subsection now recalls and appropriately adapts this construction from the case of propositional intermediate logics (see [6] for a comprehensive treatment of this propositional case for intermediate logics).

Throughout, let $\mathbf{L}$ be an intermediate logic.

Definition 9.2. A *tableau* is a tuple $\tau = (\Gamma, \Delta)$ with $\Gamma, \Delta \subseteq \mathcal{L}_J$. τ is called $\mathbf{LP}(\mathbf{L})$-consistent if

$$\Gamma \not\vdash_{\mathbf{LP}(\mathbf{L})} \phi_1 \vee \cdots \vee \phi_n$$

for any $\phi_1, \dots, \phi_n \in \Delta$. τ is called maximal if $\Gamma \cup \Delta = \mathcal{L}_J$.

Lemma 9.3 (Lindenbaum)**.** *Every* $\mathbf{LP}(\mathbf{L})$*-consistent tableau* τ *can be extended to a maximal* $\mathbf{LP}(\mathbf{L})$*-consistent tableau.*

The proof is a straightforward generalization of the propositional case (see e.g. [6]). We then can form the desired model. The main difference to Fitting's canonical model used in [9], besides the additional partial order to handle the intuitionistic implication, is the use of these tableaux instead of single maximal consistent sets as common in the study of intermediate logics (see e.g. [6]) as one can not control falsified formulas by their negation.

Definition 9.4. The *standard canonical intuitionistic Fitting model for* $\mathbf{LP}(\mathbf{L})$ is the structure $\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})) = \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \mathcal{E}^{sc}, \Vdash^{sc} \rangle$ which is defined by

(1) $\mathcal{W}^{sc} := \{\tau = (\Gamma, \Delta) \mid \tau \text{ is } \mathbf{LP}(\mathbf{L})\text{-consistent and maximal}\}$,
(2) $\tau \preceq^{sc} \tau'$ iff $\Gamma \subseteq \Gamma'$ iff $\Delta \supseteq \Delta'$,
(3) $\tau \mathcal{R}^{sc} \tau'$ iff $\Gamma^{\#} \subseteq \Gamma'$ where $\Gamma^{\#} = \{\phi \in \mathcal{L}_J \mid t : \phi \in \Gamma \text{ for some } t \in Jt\}$,
(4) $\mathcal{E}^{sc}_t(\tau) = \{\phi \mid t : \phi \in \Gamma\}$,
(5) $\Vdash^{sc} (p) = \{\tau = (\Gamma, \Delta) \mid p \in \Gamma\}$,

where $\tau = (\Gamma, \Delta)$ and $\tau' = (\Gamma', \Delta')$.

Theorem 9.5. *Let* $\mathbf{L}$ *be an intermediate logic and* $\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})) = \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \mathcal{E}^{sc}, \Vdash^{sc} \rangle$ *its canonical model. For any* $\phi \in \mathcal{L}_J$ *and any* $\tau = (\Gamma, \Delta) \in \mathcal{W}^{sc}$*:*

(1) $\phi \in \Gamma \Rightarrow (\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau) \models \phi$*;*
(2) $\phi \in \Delta \Rightarrow (\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau) \not\models \phi$*.*

Proof. The proof is a simple extension of the similar results in the propositional case which can be found in [6]. Similarly, we proceed by induction on ϕ and as the reasoning for the propositional connectives and atomic formulas given in [6] also applies here, we only consider the case for $t : \phi$ where we assume

(1) $\phi \in \Gamma' \Rightarrow (\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau') \models \phi$,
(2) $\phi \in \Delta' \Rightarrow (\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau') \not\models \phi$,

for any $\tau' = (\Gamma', \Delta') \in \mathcal{W}^{sc}$.

For (1), let $\tau = (\Gamma, \Delta) \in \mathcal{W}^{sc}$ and assume $t : \phi \in \Gamma$. Then, by definition $\phi \in \mathcal{E}^{sc}_t(\tau)$ and also for any $\tau' \in \mathcal{R}^{sc}[\tau]$: $\phi \in \Gamma'$. By induction hypothesis, we have $(\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau') \models \phi$ for any $\tau' \in \mathcal{R}^{sc}[\tau]$ and combined with $\phi \in \mathcal{E}^{sc}_t(\tau)$, we have $(\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau) \models t : \phi$.

Conversely, for (2), assume $t : \phi \in \Delta$. As τ is $\mathbf{LP}(\mathbf{L})$-consistent, we have $t : \phi \notin \Gamma$ and thus $\phi \notin \mathcal{E}^{sc}_t(\tau)$. Thus, immediately we have $(\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau) \not\models t : \phi$. □

9.2. **The main results.** We formulate the main lemma and the theorem on existence of quasi-realizations in the vein of Fitting's [9]. For this, we also introduce the following notation:

Definition 9.6. Let $\mathfrak{F}$ be a Kripke frame and let $\mathfrak{M} = \langle \mathfrak{F}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$ be a intuitionistic Fitting model over $\mathfrak{F}$. Let $x \in \mathcal{D}(\mathfrak{F})$ and let further ϕ' be an uniquely annotated modal formula. We write

(1) $(\mathfrak{M}, x) \models \langle\!\langle T, \phi'\rangle\!\rangle$ if $(\mathfrak{M}, x) \models \alpha$ for all $(T, \alpha) \in \langle\!\langle T, \phi'\rangle\!\rangle$,
(2) $(\mathfrak{M}, x) \models \langle\!\langle F, \phi'\rangle\!\rangle$ if $(\mathfrak{M}, x) \not\models \alpha$ for all $(F, \alpha) \in \langle\!\langle F, \phi'\rangle\!\rangle$.

Lemma 9.7. *Let* $\mathbf{L}$ *be an intermediate logic and let* $\mathfrak{M}^{sc} := \mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})) = \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \mathcal{E}^{sc}, \Vdash^{sc}\rangle$ *be the canonical model for* $\mathbf{LP}(\mathbf{L})$ *and define* $\mathfrak{N} := \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \Vdash^{sc}\rangle$. *For all uniquely annotated* $\phi' \in \mathcal{L}'_\Box$ *and all* $\tau \in \mathcal{W}^{sc}$:

(1) $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle T, \phi'\rangle\!\rangle \Rightarrow (\mathfrak{N}, \tau) \models (\phi')^\bullet$;
(2) $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle F, \phi'\rangle\!\rangle \Rightarrow (\mathfrak{N}, \tau) \not\models (\phi')^\bullet$.

Proof. The proof is a modification of Fitting's from [9]. We recite essential parts here for completeness. The proof proceeds by induction on the structure of ϕ'. In the following, let $\tau = (\Gamma, \Delta) \in \mathcal{W}^{sc}$.

The statement is immediate for $\bot$ and for $p \in Var$. Suppose for the induction step that ϕ', ψ' are formulas with properties (1) and (2). We omit the induction steps for $\land$ and $\lor$. For $\to$ and $\Box_n$, we give, however, the following arguments:

(i) For (1), assume that $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle T, \phi' \to \psi'\rangle\!\rangle$. Let $\tau' \in \mathcal{W}^{sc}$ be such that $\tau \preceq^{sc} \tau'$. If $(\mathfrak{M}^{sc}, \tau') \models \langle\!\langle F, \phi'\rangle\!\rangle$, then by induction hypothesis, we have $(\mathfrak{N}, \tau') \not\models (\phi')^\bullet$. If $(\mathfrak{M}^{sc}, \tau') \not\models \langle\!\langle F, \phi'\rangle\!\rangle$, then there is an α with $(F, \alpha) \in \langle\!\langle F, \phi'\rangle\!\rangle$ such that $(\mathfrak{M}^{sc}, \tau') \models \alpha$. For any β with $(T, \beta) \in \langle\!\langle T, \psi'\rangle\!\rangle$, we have $(T, \alpha \to \beta) \in \langle\!\langle T, \phi' \to \psi'\rangle\!\rangle$ by definition. By assumption of $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle T, \phi' \to \psi'\rangle\!\rangle$, we have $(\mathfrak{M}^{sc}, \tau) \models \alpha \to \beta$ and thus $(\mathfrak{M}^{sc}, \tau') \models \alpha$ implies $(\mathfrak{M}^{sc}, \tau') \models \beta$. As β was arbitrary, we have $(\mathfrak{M}^{sc}, \tau') \models \langle\!\langle T, \psi'\rangle\!\rangle$ and thus $(\mathfrak{N}, \tau') \models (\psi')^\bullet$. Combined, we have $(\mathfrak{N}, \tau) \models (\phi')^\bullet \to (\psi')^\bullet$.

Assume for (2) that $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle F, \phi' \to \psi'\rangle\!\rangle$. For any $(T, \alpha_i) \in \langle\!\langle T, \phi'\rangle\!\rangle$ and any $(F, \beta_j) \in \langle\!\langle F, \psi'\rangle\!\rangle$, we have $(F, \bigwedge_i \alpha_i \to \bigvee_j \beta_j) \in \langle\!\langle F, \phi' \to \psi'\rangle\!\rangle$. Then, for $\tau = (\Gamma, \Delta)$, the tableau

$$\mu = (\Gamma \cup \{\alpha \mid (T, \alpha) \in \langle\!\langle T, \phi'\rangle\!\rangle\}, \{\beta \mid (F, \beta) \in \langle\!\langle F, \psi'\rangle\!\rangle\})$$

is $\mathbf{LP}(\mathbf{L})$-consistent. For this, suppose not. Then, there are $\alpha_i \in \{\alpha \mid (T, \alpha) \in \langle\!\langle T, \phi'\rangle\!\rangle\}$, $\beta_j \in \{\beta \mid (F, \beta) \in \langle\!\langle F, \psi'\rangle\!\rangle\}$ such that

$$\Gamma \cup \{\alpha_1, \dots, \alpha_k\} \vdash_{\mathbf{LP}(\mathbf{L})} \beta_1 \lor \cdots \lor \beta_l$$

By the deduction theorem we have

$$\Gamma \vdash_{\mathbf{LP}(\mathbf{L})} \bigwedge_{i=1}^k \alpha_i \to \bigvee_{j=1}^l \beta_j$$

which gives $\bigwedge_{i=1}^k \alpha_i \to \bigvee_{j=1}^l \beta_j \in \Gamma$ and therefore $(\mathfrak{M}^{sc}, \tau) \models \bigwedge_{i=1}^k \alpha_i \to \bigvee_{j=1}^l \beta_j$ and this is a contradiction to $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle F, \phi' \to \psi'\rangle\!\rangle$.

As μ is consistent, there is an extension to a maximal $\mathbf{LP}(\mathbf{L})$-consistent tableau τ'. By construction, we have $\tau \preceq^{sc} \tau'$. Also, we have again by construction that

$$(\mathfrak{M}^{sc}, \tau') \models \langle\!\langle T, \phi'\rangle\!\rangle \text{ and } (\mathfrak{M}^{sc}, \tau') \models \langle\!\langle F, \psi'\rangle\!\rangle.$$

By the induction hypothesis, we have

$$(\mathfrak{N}, \tau') \models (\phi')^\bullet \text{ and } (\mathfrak{N}, \tau') \not\models (\psi')^\bullet$$

which is $(\mathfrak{N}, \tau) \not\models (\phi')^\bullet \to (\psi')^\bullet$ as $\tau \preceq^{sc} \tau'$.

(ii) For (1), suppose $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle T, \Box_n \phi'\rangle\!\rangle$. This gives by definition $(\mathfrak{M}^{sc}, \tau) \models x_n : \alpha$ for all $(T, \alpha) \in \langle\!\langle T, \phi'\rangle\!\rangle$. Thus, by definition of $\mathfrak{M}^{sc}$ we have $x_n : \alpha \in \Gamma$ and therefore $\alpha \in \Gamma^\#$. Let $\tau' = (\Gamma', \Delta') \in \mathcal{W}^{sc}$ with $\tau \mathcal{R}^{sc} \tau'$, then especially $\alpha \in \Gamma'$ by definition of $\mathcal{R}^{sc}$. As α was arbitrary, this entails

$$(\mathfrak{M}^{sc}, \tau') \models \langle\!\langle T, \phi'\rangle\!\rangle$$

and by induction hypothesis, we have $(\mathfrak{N}, \tau') \models (\phi')^\bullet$. As τ' was arbitrary, we have $(\mathfrak{N}, \tau) \models \Box(\phi')^\bullet$.

For (2), suppose $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle F, \Box_n \phi'\rangle\!\rangle$. Then, the tableau

$$\mu = (\Gamma^\#, \{\alpha \mid (F, \alpha) \in \langle\!\langle F, \phi'\rangle\!\rangle\})$$

is **LP**(**L**)-consistent. Suppose not, then there are $\gamma_i \in \Gamma^{\#}$ with

$$\{\gamma_1, \ldots, \gamma_n\} \vdash_{\mathbf{LP(L)}} \alpha_1 \vee \cdots \vee \alpha_n.$$

for some α_i with $(F, \alpha_i) \in \langle\!\langle F, \phi' \rangle\!\rangle$. As $\gamma_i \in \Gamma^{\#}$, we have $t_i : \gamma_i \in \Gamma$ for some t_i. Thus, by the lifting lemma there is a $t \in Jt$ with

$$\{t_1 : \gamma_1, \ldots, t_n : \gamma_n\} \vdash_{\mathbf{LP(L)}} t : (\alpha_1 \vee \cdots \vee \alpha_n).$$

But then, by maximality of τ we have $t : (\alpha_1 \vee \cdots \vee \alpha_n) \in \Gamma$, i.e. $(\mathfrak{M}^{sc}, \tau) \models t : (\alpha_1 \vee \cdots \vee \alpha_n)$, a contradiction to $(\mathfrak{M}^{sc}, \tau) \models \langle\!\langle F, \Box_n \phi' \rangle\!\rangle$.

As μ is consistent, it has an extension to a maximal consistent tableau τ'. Now, for this tableau τ', we have $\tau \mathcal{R}^{sc} \tau'$ by construction and we have, also by construction, that

$$(\mathfrak{M}^{sc}, \tau') \models \langle\!\langle F, \phi' \rangle\!\rangle.$$

By the induction hypothesis, we have $(\mathfrak{N}, \tau') \not\models (\phi')^{\bullet}$, and therefore by definition $(\mathfrak{N}, \tau) \not\models \Box(\phi')^{\bullet}$.

□

Theorem 9.8. *Let* **L** *be an intermediate logic. Further, let* $\mathsf{C} \in \mathsf{MFr}(\mathbf{S4}(\mathbf{L}))$. *Suppose that* $\langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc} \rangle \in \mathsf{C}$ *where* $\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})) = \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \mathcal{E}^{sc}, \Vdash^{sc} \rangle$ *is the canonical model for* **LP**(**L**).

If $\mathbf{S4}(\mathbf{L}) \vdash \phi$, *then there exists a quasi-realization* $\alpha_1 \vee \cdots \vee \alpha_n$ *of* ϕ *with* $\mathbf{LP}(\mathbf{L}) \vdash \alpha_1 \vee \cdots \vee \alpha_n$.

Proof. Suppose $\nvdash_{\mathbf{LP(L)}} \alpha_1 \vee \cdots \vee \alpha_n$ for all $(F, \alpha_i) \in \langle\!\langle F, \phi' \rangle\!\rangle$. Then the tableau

$$(\emptyset, \{\alpha \mid (F, \alpha) \in \langle\!\langle F, \phi' \rangle\!\rangle\})$$

is **LP**(**L**)-consistent. Thus, it extends to a maximal consistent tableau $\tau = (\Gamma, \Delta) \in \mathcal{W}^{sc}$. As $\{\alpha \mid (F, \alpha) \in \langle\!\langle F, \phi' \rangle\!\rangle\} \subseteq \Delta$, we have $(\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})), \tau) \models \langle\!\langle F, \phi' \rangle\!\rangle$. By the previous lemma, we have $(\mathfrak{N}, \tau) \not\models (\phi')^{\bullet}$ for $\mathfrak{N} := \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \Vdash^{sc} \rangle$. As $\langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc} \rangle \in \mathsf{C}$, we have by the choice of C that $\mathbf{S4}(\mathbf{L}) \nvdash \phi$. □

10. Unified Realization for Intermediate Modal Logics

In this section, we adapt Fittings algorithm for the construction of realizations from quasi-realizations to the intermediate case, culminating in a realization theorem for intermediate modal logics. This amounts, modulo some modifications in the $\to$-case, to verifying that Fitting's construction and proof from [9] also works in **J**(**IPC**) which is the minimal intermediate justification logic defined by $\mathbf{J}_{TCS_{\mathbf{J}_0(\mathbf{IPC})}}(\mathbf{IPC})$ where

$$\mathbf{J}_0(\mathbf{IPC}) = \overline{\mathbf{IPC}} + (J) + (+).$$

is defined as before (now in the case of **IPC**) and $TCS_{\mathbf{J}_0(\mathbf{IPC})}$ is the total constant specification for that logic.

Following [9], we introduce a special notation for the following algorithm transforming Quasi-Realizations into Realizations.

Definition 10.1. Let $\phi' \in \mathcal{L}'_{\Box}$ and $\Gamma \cup \{\psi\} \subseteq \mathcal{L}_J$ where Γ is finite. Let σ be a justification substitution. We write:

(1) $\Gamma \xrightarrow{T\phi'} (\psi, \sigma)$ if (1) $\{T\} \times \Gamma \subseteq \langle\!\langle T, \phi' \rangle\!\rangle$, (2) $(T, \psi) \in [\![T, \phi']\!]$ and
$$(3)\ \mathbf{J}(\mathbf{IPC}) \vdash \psi \to \left(\bigwedge \Gamma\right) \sigma;$$

(2) $\Gamma \xrightarrow{F\phi'} (\psi, \sigma)$ if (1) $\{F\} \times \Gamma \subseteq \langle\!\langle F, \phi' \rangle\!\rangle$, (2) $(F, \psi) \in [\![F, \phi']\!]$ and
$$(3)\ \mathbf{J}(\mathbf{IPC}) \vdash \left(\bigvee \Gamma\right) \sigma \to \psi.$$

The following algorithm is a slight modification of that Fitting from [9].

Algorithm 10.2. Atomic Case: *The atomic propositions have a trivial realization through the empty justification substitution* ε*:*

$$\{p\} \xrightarrow{Tp} (p, \varepsilon) \qquad \{p\} \xrightarrow{Fp} (p, \varepsilon)$$
$$\{\bot\} \xrightarrow{T\bot} (\bot, \varepsilon) \qquad \{\bot\} \xrightarrow{F\bot} (\bot, \varepsilon)$$

$T\wedge$ **Case:**

$$\frac{\{\alpha_1, \ldots, \alpha_k\} \xrightarrow{F\phi'} (\chi, \sigma_{\phi'}) \quad \{\beta_1, \ldots, \beta_k\} \xrightarrow{T\psi'} (\xi, \sigma_{\psi'})}{\{\alpha_1 \wedge \beta_1, \ldots, \alpha_k \wedge \beta_k\} \xrightarrow{T\phi' \wedge \psi'} ((\chi \wedge \xi)\sigma_{\phi'}\sigma_{\psi'}, \sigma_{\phi'}\sigma_{\psi'})}$$

$F\wedge$ **Case:**

$$\frac{\{\alpha_1,\dots,\alpha_k\} \xrightarrow{T\phi'} (\chi,\sigma_{\phi'}) \quad \{\beta_1,\dots,\beta_k\} \xrightarrow{F\psi'} (\xi,\sigma_{\psi'})}{\{\alpha_1\wedge\beta_1,\dots,\alpha_k\wedge\beta_k\} \xrightarrow{F\phi'\wedge\psi'} ((\chi\wedge\xi)\sigma_{\phi'}\sigma_{\psi'},\sigma_{\phi'}\sigma_{\psi'})}$$

$T\vee$ **Case:**

$$\frac{\{\alpha_1,\dots,\alpha_k\} \xrightarrow{F\phi'} (\chi,\sigma_{\phi'}) \quad \{\beta_1,\dots,\beta_k\} \xrightarrow{T\psi'} (\xi,\sigma_{\psi'})}{\{\alpha_1\vee\beta_1,\dots,\alpha_k\vee\beta_k\} \xrightarrow{T\phi'\vee\psi'} ((\chi\vee\xi)\sigma_{\phi'}\sigma_{\psi'},\sigma_{\phi'}\sigma_{\psi'})}$$

$F\vee$ **Case:**

$$\frac{\{\alpha_1,\dots,\alpha_k\} \xrightarrow{T\phi'} (\chi,\sigma_{\phi'}) \quad \{\beta_1,\dots,\beta_k\} \xrightarrow{F\psi'} (\xi,\sigma_{\psi'})}{\{\alpha_1\vee\beta_1,\dots,\alpha_k\vee\beta_k\} \xrightarrow{F\phi'\vee\psi'} ((\chi\vee\xi)\sigma_{\phi'}\sigma_{\psi'},\sigma_{\phi'}\sigma_{\psi'})}$$

$T\to$ **Case:**

$$\frac{\{\alpha_1,\dots,\alpha_k\} \xrightarrow{F\phi'} (\chi,\sigma_{\phi'}) \quad \{\beta_1,\dots,\beta_k\} \xrightarrow{T\psi'} (\xi,\sigma_{\psi'})}{\{\alpha_1\to\beta_1,\dots,\alpha_k\to\beta_k\} \xrightarrow{T\phi'\to\psi'} (\chi\sigma_{\psi'}\to\xi\sigma_{\phi'},\sigma_{\phi'}\sigma_{\psi'})}$$

$F\to$ **Case:**

$$\frac{\Gamma_1\cup\dots\cup\Gamma_k \xrightarrow{T\phi'} (\chi,\sigma_{\phi'}) \quad \Delta_1\cup\dots\cup\Delta_k \xrightarrow{F\psi'} (\xi,\sigma_{\psi'})}{\{\bigwedge\Gamma_1\to\bigvee\Delta_1,\dots,\bigwedge\Gamma_k\to\bigvee\Delta_k\} \xrightarrow{F\phi'\to\psi'} (\chi\sigma_{\psi'}\to\xi\sigma_{\phi'},\sigma_{\phi'}\sigma_{\psi'})}$$

$T\Box$ **Case:**

$$\frac{\{\alpha_1,\dots,\alpha_k\} \xrightarrow{T\phi'} (\chi,\sigma_{\phi'})}{\{x_n:\alpha_1,\dots,x_n:\alpha_k\} \xrightarrow{T\Box_n\phi'} (x_n:\chi\sigma,\sigma_{\phi'}\sigma)}$$

with $\mathbf{J}(\mathbf{IPC})\vdash t_i : (\chi\to\alpha_i\sigma_{\phi'})$ *and* $\sigma(x_n)=[s\cdot x_n]$, $\sigma(x_k)=x_k$ *for* $k\neq n$ *where* $s=[t_1+\dots+t_n]$.

$F\Box$ **Case:**

$$\frac{\Gamma_1\cup\dots\cup\Gamma_k \xrightarrow{F\phi'} (\chi,\sigma_{\phi'})}{\{t_1:\bigvee\Gamma_1,\dots,t_k:\bigvee\Gamma_k\} \xrightarrow{F\Box_n\phi'} (t\sigma_{\phi'}:\chi,\sigma_{\phi'})}$$

with $\mathbf{J}(\mathbf{IPC})\vdash u_i : (\bigvee\Gamma_i\sigma_{\phi'}\to\chi)$ *and* $t=[[u_1\cdot t_1]+\dots+[u_k\cdot t_k]]$.

Theorem 10.3. *Let $\phi'\in\mathcal{L}'_\Box$ and let $\Gamma\subseteq\mathcal{L}_J$ be nonempty and finite. Then:*

(1) If $\{T\}\times\Gamma\subseteq\langle\!\langle T,\phi'\rangle\!\rangle$, there are $\psi\in\mathcal{L}_J$ and a justification substitution σ such that $\Gamma\xrightarrow{T\phi'}(\psi,\sigma)$.

(2) If $\{F\}\times\Gamma\subseteq\langle\!\langle F,\phi'\rangle\!\rangle$, there are $\psi\in\mathcal{L}_J$ and a justification substitution σ such that $\Gamma\xrightarrow{F\phi'}(\psi,\sigma)$.

Proof. By recursion on ϕ', using Algorithm 10.2, we construct the desired pair (ψ,σ). Note, that the T and F rules of the algorithm preserve $\{T\}\times\Gamma\subseteq\langle\!\langle T,\phi'\rangle\!\rangle$ or $\{F\}\times\Gamma\subseteq\langle\!\langle F,\phi'\rangle\!\rangle$, respectively.

The cases for $T\to$, $T\Box$, $F\Box$ as well as the atomic case were handled in [9] and the arguments also apply here. We omit the cases for $\wedge$ and $\vee$ as they are quite elementary. We give the one of $T\to$ in some detail. The main difference is however in the case for $F\to$, as we have modified the definition of $\langle\!\langle F,\phi'\to\psi'\rangle\!\rangle$.

($F\to$): Assume that we have

(i) $\Gamma_1\cup\dots\cup\Gamma_k\xrightarrow{T\phi'}(\chi,\sigma_{\phi'})$,

(ii) $\Delta_1\cup\dots\cup\Delta_k\xrightarrow{F\psi'}(\xi,\sigma_{\psi'})$.

Then we have

$$\{\bigwedge\Gamma_1\to\bigvee\Delta_1,\dots,\bigwedge\Gamma_k\to\bigvee\Delta_k\}\subseteq\langle\!\langle F,\phi'\to\psi'\rangle\!\rangle.$$

We have, by the requirements on substitutions (similarly as in [9]), that $\sigma_{\phi'}\sigma_{\psi'}=\sigma_{\psi'}\sigma_{\phi'}$ and by (i) and (ii), we have

$$(T,\chi\sigma_{\psi'})\in[\![T,\phi']\!] \text{ and } (F,\xi\sigma_{\phi'})\in[\![F,\psi']\!]$$

as we have $(T,\chi)\in[\![T,\phi']\!]$ and $[\![\cdot]\!]$ is closed under substitutions (and similarly for $[\![F,\psi']\!]$). Thus, we have

$$(F,\chi\sigma_{\psi'}\to\xi\sigma_{\phi'})\in[\![F,\phi'\to\psi']\!].$$

Now, we obtain

$$\mathbf{J}(\mathbf{IPC})\vdash\bigvee_{i=1}^{k}\left(\bigwedge\Gamma_i\to\bigvee\Delta_i\right)\sigma_{\phi'}\sigma_{\psi'}\to(\chi\sigma_{\psi'}\to\xi\sigma_{\phi'})$$

as by (i), we have

$$\mathbf{J}(\mathbf{IPC})\vdash\chi\sigma_{\psi'}\to\left(\bigwedge_{i=1}^{k}\bigwedge\Gamma_i\right)\sigma_{\phi'}\sigma_{\psi'}$$

and by (ii), we have

$$\mathbf{J}(\mathbf{IPC}) \vdash \left(\bigvee_{i=1}^{k} \bigvee \Delta_i\right) \sigma_{\psi'}\sigma_{\phi'} \to \xi\sigma_{\phi'}.$$

As $\sigma_{\phi'}\sigma_{\psi'} = \sigma_{\psi'}\sigma_{\phi'}$ and by the other properties of substitutions, we obtain the claim by utilizing the following validity of intuitionistic logic:

$$\mathbf{J}(\mathbf{IPC}) \vdash \bigwedge_{i=1}^{k} \bigwedge \Gamma_i \wedge \bigvee_{i=1}^{k} \left(\bigwedge \Gamma_i \to \bigvee \Delta_i\right) \to \bigvee_{i=1}^{k} \bigvee \Delta_i.$$

$(T \to)$: Suppose that

(i) $\{\alpha_1, \dots, \alpha_k\} \xrightarrow{F\phi'} (\chi, \sigma_{\phi'})$,

(ii) $\{\beta_1, \dots, \beta_k\} \xrightarrow{T\psi'} (\xi, \sigma_{\psi'})$.

Then we naturally have

$$\{(T, \alpha_1 \to \beta_1), \dots, (T, \alpha_k \to \beta_k)\} \subseteq \langle\!\langle T, \phi' \to \psi' \rangle\!\rangle.$$

As before, one shows $\sigma_{\phi'}\sigma_{\psi'} = \sigma_{\psi'}\sigma_{\phi'}$ and also similarly one shows

$$(T, \chi\sigma_{\psi'} \to \xi\sigma_{\phi'}) \in [\![T, \phi' \to \psi']\!].$$

Now, have by (i) that

$$\mathbf{J}(\mathbf{IPC}) \vdash \left(\bigvee_{i=1}^{k} \alpha_k\right) \sigma_{\phi'} \to \chi$$

and by (ii):

$$\mathbf{J}(\mathbf{IPC}) \vdash \xi \to \left(\bigwedge_{i=1}^{k} \beta_k\right) \sigma_{\psi'}$$

We obtain

$$\mathbf{J}(\mathbf{IPC}) \vdash (\chi\sigma_{\psi'} \to \xi\sigma_{\phi'}) \to \bigwedge_{i=1}^{k} (\alpha_i \to \beta_i)\sigma_{\phi'}\sigma_{\psi'}$$

by utilizing the following validity of intuitionistic logic:

$$\mathbf{J}(\mathbf{IPC}) \vdash \left(\bigvee_{i=1}^{k} \alpha_i \to \bigwedge_{i=1}^{k} \beta_i\right) \to \bigwedge_{i=1}^{k} (\alpha_i \to \beta_i).$$

□

Theorem 10.4. *Let* $\mathbf{L}$ *be an intermediate logic and and let* $\mathsf{C} \in \mathsf{MFr}(\mathbf{S4}(\mathbf{L}))$. *Let* $\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})) = \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \mathcal{E}^{sc}, \Vdash^{sc} \rangle$ *be the canonical model of* $\mathbf{LP}(\mathbf{L})$ *and suppose that* $\langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc} \rangle \in \mathsf{C}$.

If $\mathbf{S4}(\mathbf{L}) \vdash \phi$, *then there exists a* $\psi \in \mathcal{L}_J$ *with* $(F, \psi) \in [\![F, \phi']\!]$ *and* $\mathbf{LP}(\mathbf{L}) \vdash \psi$.

Proof. By Theorem 9.8, $\mathbf{S4}(\mathbf{L}) \vdash \phi$ implies that there exist $(F, \alpha_i) \in \langle\!\langle F, \phi' \rangle\!\rangle$, for some annotation ϕ' of ϕ, such that $\mathbf{LP}(\mathbf{L}) \vdash \alpha_1 \vee \dots \vee \alpha_n$.

Now, by Theorem 10.3, there is a σ and a $(F, \psi) \in [\![F, \phi']\!]$ such that $\{\alpha_i \mid i \leq n\} \xrightarrow{F\phi'} (\psi, \sigma)$. By definition, $\mathbf{J}(\mathbf{IPC}) \vdash (\alpha_1 \vee \dots \vee \alpha_n)\sigma \to \psi$ and thus $\mathbf{LP}(\mathbf{L}) \vdash \psi$ as Lemma 8.3 implies $\mathbf{LP}(\mathbf{L}) \vdash (\alpha_1 \vee \dots \vee \alpha_n)\sigma$. □

11. Conclusion

We again first want to remark that although the main focus of the paper were the justification logics $\mathbf{LP}_{CS}(\mathbf{L})$, the results presented here extend to intermediate version of the usual justification logics **J**, **JT** and **J4** as defined before. In that vein, the completeness theorems proved in this paper show that a unified completeness theorem stands behind all previously established completeness results from the literature for justification logics with intermediate base logic, including the classical, intuitionistic and Gödel logic cases discussed before. The results contained in Theorems 4.4, 4.7 and 4.10 in particular imply the known completeness theorems of Mkrtychev [23], Fitting [8] as well as Lehmann and Studer [20] by considering classical logic **CPC** and using completeness w.r.t. the two-valued Boolean algebra. They further imply the completeness results for Gödel justification logics contained in [27] by working over the infinite-valued Gödel-Dummett logic **GD** [7] and using completeness w.r.t. the Heyting algebra over $[0, 1]$ with the usual order.

Similarly, the completeness results for the intuitionistic semantics imply the previously known completeness results: **IPC** is complete w.r.t. the class of all intuitionistic Kripke frames IF (going back to Kripke's work [15]). As IF is closed under principal subframes, Lemma 5.5 implies completeness of $\mathbf{LP}_{CS}(\mathbf{IPC})$ w.r.t. the respective intuitionistic model classes over these frames which implies the results of [21] where the authors considered models for $\mathbf{LP}_{CS}(\mathbf{IPC})$ which are essentially the same as the models introduced here.

Moreover, the completeness theorems established here do not only offer a uniformity w.r.t. the intermediate base logic but also, and more importantly, w.r.t. the model classes of the resulting justification logic as they allow one to transfer a propositional completeness result for the base logic w.r.t. classes of algebras or Kripke frames to a corresponding completeness result for the extension by justification axioms which in particular only requires models using algebras or frames from these classes. Key to this is of course the strong completeness assumption of the underlying propositional logic and, in particular, that of global completeness in the case of Kripke frames.

In a similar sense as with the completeness theorem, the realization theorem proved here shows that there is a unified result behind many of the previous realization theorems from the literature: using a usual canonical model construction similar to that of Section 9.1, it is straightforward to show that

(1) $\mathbf{S4}(\mathbf{IPC})$ is strongly complete w.r.t. to intuitionistic modal Kripke models with reflexive and transitive $\mathcal{R}$,
(2) $\mathbf{S4}(\mathbf{GD})$ is strongly complete w.r.t. to intuitionistic modal Kripke models with reflexive and transitive $\mathcal{R}$ and connected order $\leq$,
(3) $\mathbf{S4}(\mathbf{CPC})$ is strongly complete w.r.t. to intuitionistic modal Kripke models with reflexive and transitive $\mathcal{R}$ and trivial order $\leq$ (i.e. $x \leq y$ iff $x = y$).

Now, in analogy to the corresponding result from [6] for intermediate propositional and classical modal logics (see again Theorem 5.16 there): the order $\preceq^{sc}$ of the canonical model $\mathfrak{M}^{sc}(\mathbf{LP}(\mathbf{L})) = \langle \mathcal{W}^{sc}, \preceq^{sc}, \mathcal{R}^{sc}, \mathcal{E}^{sc}, \Vdash^{sc} \rangle$ of $\mathbf{LP}(\mathbf{L})$ is connected if $\mathbf{L} = \mathbf{GD}$ and trivial if $\mathbf{L} = \mathbf{CPC}$.

By the uniform realization results, this in particular implies the following realization results from the literature (in a non-constructive way, however):

(1) $\mathbf{LP}(\mathbf{IPC})$ realizes $\mathbf{S4}(\mathbf{IPC})$ ([21]),
(2) $\mathbf{LP}(\mathbf{GD})$ realizes $\mathbf{S4}(\mathbf{GD})$ ([28]),
(3) $\mathbf{LP}(\mathbf{CPC})$ realizes $\mathbf{S4}(\mathbf{CPC})$ ([1, 2]).

In the corresponding modal logics, we have only considered intermediate modal logics with a single modality $\Box$. As the dual $\Diamond$ is not definable in intermediate cases, it would also require a separate treatment in the justification contexts. There has been a recent work [18] on treating a non-dual $\Diamond$ in the context of constructive modal logics explicitly in the style of justification logics and it would be interesting to see if and how similar methods as exhibited in this paper could be applied in that context.

Lastly, it may be interesting to study the use of these semantics introduced here for the intermediate justification logics for providing semantics for normal intermediate modal logics. In [28] it was shown that there are specific instances of normal intermediate modal logics which do not enjoy a completeness theorem w.r.t. a natural algebraic generalization of the usual Kripke semantics but where semantics for intermediate justification logics (concretely Gödel justification logics) have been used to provide an alternative complete semantics in those cases. It would be interesting to see if this applies to other situations as well.

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