

ON INFINITARY GÖDEL LOGICS

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ABSTRACT. We study propositional and first-order Gödel logics over infinitary languages which are motivated semantically by corresponding interpretations into the unit interval $[0, 1]$. We provide infinitary Hilbert-style calculi for the particular (propositional and first-order) cases with con-/disjunctions of countable length and prove corresponding completeness theorems by extending the usual Lindenbaum-Tarski construction to the infinitary case for a respective algebraic semantics via complete linear Heyting algebras. We provide infinitary hypersequent calculi and prove corresponding cut-elimination theorems in the Schütte-Tait-style. Initial observations are made regarding truth-value sets other than $[0, 1]$.

1. INTRODUCTION

Infinitary logics in a classical setting go back to [23, 31, 35] and over time became influential in various areas of mathematical logic like (finite) model theory, set theory and also formal arithmetic, among others. Model theoretically, they pose an interesting challenge since the usual propositional and first-order properties (like e.g. compactness) become more intertwined with set-theoretic principles (like e.g. large cardinal axioms).

We study infinitary extensions of Gödel logics. In the finitary setting, Gödel logics arose historically from a sequence of propositional finite-valued logics given by Gödel [17] to show that intuitionistic logic does not have a finite characteristic matrix. These were extended to an infinite-valued variant by Dummett [15] and the whole collection is today especially studied in the context of intermediate logics. Further, Gödel logics have been characterized as one of three main instances of t-norm based fuzzy logics by Hájek [18]. First-order versions were first described by Horn [21] and later rediscovered by Takeuti and Titani [34] under the name of intuitionistic fuzzy logics. The infinitary versions studied here assume a similar position among both the infinitary intermediate and infinitary fuzzy logics and the present work is thus, in that way, also a particular case study of these classes. For that purpose, Gödel logics pose an especially interesting case as they, in the intermediate context, are logics with many classical properties but which, at the same time, are distinct enough from classical or intuitionistic logic to still pose interesting methodological challenges for the adaption of well-known results. An example of this phenomenon in the infinitary setting is the work [1] by Aguilera where he studies analogues of the compactness results for classical infinitary logics [20, 24] in a Gödel setting where, although the classical results stay true modulo appropriate reformulations, the methods of Skolem functions used classically had to be reformulated using a certain theory of fuzzy ultraproducts.

Two interesting avenues not explored here but worth mentioning (also to motivate potential further work) are the following: For one, the connection between infinitary Gödel logics and Kripke frames, where in particular it may be interesting to see how the results on general linearly ordered Kripke frames and their connection to Gödel logics explored in [13] for the finitary setting extend to the infinitary setting, in particular in the light of the results on large Kripke frames from [26]. For another, one may investigate the use of infinitary connectives for axiomatizing extensions of propositional and first-order Gödel logics. For example, in the context of quantified propositional Gödel logics [6, 7, 9], one relies on the use of a “jump” operator \bigcirc for decidability (see [6]) which may be definable using the infinitary connectives, potentially providing novel axiomatizations.¹

By now, the work [1] is the only paper on infinitary Gödel logics and many interesting problems arising from generalizations of the classical case have remained open, like the study of propositional variants, the development of infinitary calculi and appropriate completeness theorems in the propositional and first-order case as well as infinitary structural proof theory, among others. We study all these previously named topics (at varying depth) for the instances with conjunctions and disjunctions of countable length and finitely many quantifiers, like in the classical [23] and intuitionistic cases [22, 27] (see also the recent [36]). Concretely, this paper is organized as follows: Section 2 introduces the language, semantics and a prospective Hilbert-style proof calculus for the propositional case. Section 3 provides a similar introduction for the first-order case. Section 4 introduces the algebraic semantics for these languages based on Heyting algebras (where some essential properties and results are recalled). Completeness results w.r.t. the algebraic and, through that, the standard semantics are proved

Date: April 10, 2022.

Key words and phrases. Gödel Logic, Infinitary Logic, Completeness, Cut-Elimination.

¹As pointed out to us by one of the anonymous referees.

in section 5 and in section 6, we introduce hypersequent calculi for the propositional and first-order variants and prove a cut-elimination theorem. At last, in section 7 we motivate and present some open questions and discuss other choices for the set of truth values where we derive a range of further completeness results from the respective completeness result for the set $[0, 1]$ considered before.

2. PROPOSITIONAL INFINITARY GÖDEL LOGICS

2.1. Syntax and Fragments. Let κ be any cardinal number. The *infinitary propositional language associated with κ* is given by

$$\mathcal{L}_\kappa : \phi ::= \perp \mid x \mid (\phi \rightarrow \phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \bigwedge \Phi \mid \bigvee \Phi$$

where we have $x \in \text{Var}_\kappa := \{x_\lambda \mid \lambda \in \kappa\}$ and Φ is a set of formulas of size $< \kappa$. We also call \perp and x atomics. For the other usual operators, we define

- (1) $\neg\phi := \phi \rightarrow \perp$,
- (2) $\top := \neg\perp$,
- (3) $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

Given some formula ϕ , we denote the set of all subformulas (including ϕ) by $\text{sub}(\phi)$ and the set of variables of ϕ by $\text{var}(\phi)$. Both of these naturally extend to sets Γ .

We mostly deal with the special case of $\kappa = \omega_1$ and in that context, we will mostly write $\bigwedge_{i \in \omega} \phi_i$ for $\bigwedge\{\phi_i \mid i \in \omega\}$ and $\bigvee_{i \in \omega} \phi_i$ for $\bigvee\{\phi_i \mid i \in \omega\}$ where $(\phi_i)_{i \in \omega}$ is a countable family of formulas.

Related to that particular instance of $\kappa = \omega_1$, we will also need the notion of a *fragment*. These fragments are (possibly countable) sublanguages of \mathcal{L}_{ω_1} for which a Lindenbaum-Tarski construction is, nevertheless, still possible and they form a cornerstone of the proof of the completeness theorem. While these fragments, in particular the notation \mathcal{L}_A , originate from the connection of (classical) infinitary logic with admissible sets in the sense of Barwise [12], we only need and use the following syntactic definition, in similarity to Nadel [27] in the context of infinitary intuitionistic logic.

Definition 2.1. A (*distributive*) *fragment of \mathcal{L}_{ω_1}* is a set $\mathcal{L}_A \subseteq \mathcal{L}_{\omega_1}$ such that

- (1) $\perp \in \mathcal{L}_A$,
- (2) $\phi \in \mathcal{L}_A$ implies $\text{sub}(\phi) \subseteq \mathcal{L}_A$,
- (3) $\phi, \psi \in \mathcal{L}_A$ implies $\phi \circ \psi \in \mathcal{L}_A$ for $\circ \in \{\wedge, \vee, \rightarrow\}$,
- (4) $\phi, \bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigwedge_{i \in \omega} (\phi \rightarrow \phi_i) \in \mathcal{L}_A$,
- (5) $\phi, \bigvee_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigwedge_{i \in \omega} (\phi_i \rightarrow \phi) \in \mathcal{L}_A$,
- (6) $\phi, \bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigwedge_{i \in \omega} (\phi \vee \phi_i) \in \mathcal{L}_A$.

The important kind of fragments will be countable ones. In particular, we will consider the smallest fragments containing some set of formulas.

Lemma 2.2. *For any $\Gamma \subseteq \mathcal{L}_{\omega_1}$, there is a smallest (w.r.t. \subseteq) distributive fragment $\text{frag}(\Gamma)$ such that $\Gamma \subseteq \text{frag}(\Gamma)$. If Γ is countable, then $\text{frag}(\Gamma)$ is also countable.*

2.2. The Standard Semantics and \mathcal{G}_κ . We now introduce the standard semantics for the language \mathcal{L}_κ and the resulting semantic consequence relation. This standard semantics naturally extends the usual finitary standard semantics for propositional Gödel logics. For Gödel logics, being one of the prime examples of many-valued logics, the first important parameter in that context is that of the truth-value set. As mentioned before, we fix this to be $[0, 1]$ for the major part of the paper and discuss other choices only in part at the end.

Definition 2.3. An *\mathcal{L}_κ -Gödel-evaluation* is a function $v : \mathcal{L}_\kappa \rightarrow [0, 1]$ such that

- (1) $v(\perp) = 0$,
- (2) $v(\phi \wedge \psi) = \min\{v(\phi), v(\psi)\}$,
- (3) $v(\phi \vee \psi) = \max\{v(\phi), v(\psi)\}$,
- (4) $v(\phi \rightarrow \psi) = v(\phi) \Rightarrow v(\psi)$ where $x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$
- (5) $v(\bigwedge \Phi) = \inf\{v(\phi) \mid \phi \in \Phi\}$,
- (6) $v(\bigvee \Phi) = \sup\{v(\phi) \mid \phi \in \Phi\}$,

for any $\Phi \cup \{\phi, \psi\} \subseteq \mathcal{L}_\kappa$.

Given a set of formulas Γ , we write $v[\Gamma] := \{v(\gamma) \mid \gamma \in \Gamma\}$. The derived notion of semantics consequence is then defined as follows: for $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\kappa$, we write $\Gamma \models_{\mathcal{G}_\kappa} \phi$ if $v[\Gamma] \subseteq \{1\}$ implies $v(\phi) = 1$ for any \mathcal{L}_κ -Gödel-evaluation v .

We call the set of consequences $\Gamma \models_{\mathcal{G}_\kappa} \phi$ the κ -*infinitary Gödel logic* and denote it by \mathcal{G}_κ . For the particular case of \mathcal{G}_{ω_1} , a main part of the paper is devoted to the study of various proof theoretic formalisms for capturing that semantic consequence and we thus continue by introducing the relevant Hilbert-style calculus used later in a corresponding completeness proof.

Before that, we also want to note that there is a different definition of consequence common in the context of Gödel logics, which we may define by

$$\Gamma \models_{\mathcal{G}_\kappa}^{\leq} \phi \text{ if } \inf v[\Gamma] \leq v(\phi) \text{ for any evaluation } v.$$

In a finitary context, it can be easily seen that \models^{\leq} is equivalent to \models and we will show a similar statement (for countable sets of premises) here later on (see Proposition 4.10).

2.3. A Proof Calculus for $\kappa = \omega_1$. The proof calculus for \mathcal{G}_{ω_1} which we introduce, denoted by \mathcal{G}_{ω_1} , is a straightforward combination of a proof calculus for propositional infinitary intuitionistic logic with the (pre-)linearity scheme

$$(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi).$$

To be concrete, following [37], we consider the following system of axioms and rules (*IL*) for intuitionistic propositional logic:

$$\begin{aligned} (\wedge\text{-Ax}): & \begin{cases} \phi \wedge \psi \rightarrow \phi; \phi \wedge \psi \rightarrow \psi; \\ \phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)); \end{cases} \\ (\vee\text{-Ax}): & \begin{cases} \phi \rightarrow \phi \vee \psi; \psi \rightarrow \phi \vee \psi; \\ (\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \vee \psi \rightarrow \chi)); \end{cases} \\ (\rightarrow\text{-Ax}): & \begin{cases} \phi \rightarrow (\psi \rightarrow \phi); \\ (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)); \end{cases} \\ (\perp\text{-Ax}): & \perp \rightarrow \phi. \end{aligned}$$

Then, we define:

The Calculus \mathcal{G}_{ω_1} .

(*IL*): the complete set of axioms for propositional intuitionistic logic considered above;

(*GL*): $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$;

($\omega\vee$): $\phi_j \rightarrow \bigvee_{i \in \omega} \phi_i$, ($j \in \omega$);

($\omega\wedge$): $\bigwedge_{i \in \omega} \phi_i \rightarrow \phi_j$, ($j \in \omega$);

(*MP*): from $\phi \rightarrow \psi$ and ϕ , infer ψ ;

($R\omega$)₁: from $\phi_i \rightarrow \psi$ for all $i \in \omega$, infer $\bigvee_{i \in \omega} \phi_i \rightarrow \psi$;

($R\omega$)₂: from $\phi \rightarrow \psi_i$ for all $i \in \omega$, infer $\phi \rightarrow \bigwedge_{i \in \omega} \psi_i$.

Further, we define the extension $\mathcal{G}_{\omega_1}^D$ by adding the axiom scheme

$$(D) \quad \bigwedge_{i \in \omega} (\phi \vee \psi_i) \rightarrow \left(\phi \vee \bigwedge_{i \in \omega} \psi_i \right),$$

expressing the distributivity of the infinitary operations, to the above calculus.

A proof in \mathcal{G}_{ω_1} (or $\mathcal{G}_{\omega_1}^D$) of some ϕ from some assumptions Γ is any function $f : \alpha + 1 \rightarrow \mathcal{L}_{\omega_1}$ where $\alpha < \omega_1$ as well as $f(\alpha) = \phi$ and such that any $f(\beta)$ is either

- (1) an instance of an axiom scheme,
- (2) an element of Γ ,
- (3) the result of (*MP*), ($R\omega$)₁ or ($R\omega$)₂ with assumptions $f(\gamma)$ where $\gamma < \beta$.

We write $\Gamma \vdash_{\mathcal{G}_{\omega_1}} \phi$ (or $\Gamma \vdash_{\mathcal{G}_{\omega_1}^D} \phi$, respectively) if there is such a proof.

Relative to some fragment \mathcal{L}_A as defined above, we also introduce a restricted notion of derivation: for $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$, we write $\Gamma \vdash_{\mathcal{G}_{\omega_1}(\mathcal{L}_A)} \phi$ (or $\Gamma \vdash_{\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)} \phi$, respectively) if there is a proof f with $\text{img}(f) \subseteq \mathcal{L}_A$.

Note that $\mathcal{G}_{\omega_1}(\mathcal{L}_A)$, and thus also $\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)$, have the classical Deduction Theorem.

Lemma 2.4. *For any $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_A$, we have*

$$\Gamma \cup \{\phi\} \vdash_{\mathcal{G}_{\omega_1}(\mathcal{L}_A)} \psi \text{ iff } \Gamma \vdash_{\mathcal{G}_{\omega_1}(\mathcal{L}_A)} \phi \rightarrow \psi.$$

The same holds for $\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)$.

The classical proof (see, e.g., [25]) immediately goes through due to the axioms (\rightarrow -Ax) and the modus ponens rule.

3. FIRST-ORDER INFINITARY GÖDEL LOGICS

For the first-order variant, we assume a standard underlying first-order signature σ consisting of any number of predicate symbols P and functions symbols f . For any such given symbol, we write $\text{ar}(P)$ or $\text{ar}(f)$, respectively, for its arity (which is assumed to be finite).

We construct an infinitary language corresponding to cardinals $\kappa \geq \lambda$ as it is usually done classically as well: assume a set of variables of size κ , given by Var_κ as before, and in that context denote the set of terms over σ and Var_κ by $\mathcal{T}_\kappa(\sigma)$. The *infinitary language over σ associated with κ, λ* is then given by

$$\mathcal{L}_{\kappa, \lambda}(\sigma) : \phi ::= \perp \mid P(t_1, \dots, t_n) \mid (\phi \rightarrow \phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \bigwedge \Phi \mid \bigvee \Phi \mid \exists X \phi \mid \forall X \phi$$

where $X \subseteq \text{Var}_\kappa$ is a set of size $< \lambda$, $P \in \sigma$ is a predicate symbol with $\text{ar}(P) = n$, $t_1, \dots, t_n \in \mathcal{T}_\kappa(\sigma)$ and Φ is a set of formulas of size $< \kappa$. We drop the σ if the context is clear or if the choice is arbitrary. Similar to before, we also call \perp and all instances of $P(t_1, \dots, t_n)$ atomics.

We write $\text{free}(\phi)$ for the set of free variables of ϕ and $\text{var}(\phi)$ for the set of variables of ϕ , free or bound. We call ϕ closed if $\text{free}(\phi) = \emptyset$. As before, we write $\text{sub}(\phi)$ for the set of subformulas of ϕ where $\text{sub}(QX\phi) := \{\phi\} \cup \text{sub}(\phi)$ for $Q \in \{\exists, \forall\}$. Further, we write $\sigma(\phi)$ for the set of function and predicate symbols occurring in ϕ . These notions straightforwardly extend to sets Γ and we use the same notation there.

Again, the countable case $\mathcal{L}_{\omega_1, \omega}$ with finite quantifiers will be of particular interest, especially in the context of a completeness theorem, later on. In that case, we consider the existential and universal quantifiers to just quantify one variable at a time and write

$$\exists x \phi \text{ or } \forall x \phi$$

for $x \in \text{Var}_{\omega_1}$ as usual in that case. Further, we again write $\bigwedge_{i \in \omega} \phi_i$ for $\bigwedge \{\phi_i \mid i \in \omega\}$ and $\bigvee_{i \in \omega} \phi_i$ for $\bigvee \{\phi_i \mid i \in \omega\}$ where $(\phi_i)_{i \in \omega}$ is a countable family of formulas.

On $\mathcal{L}_{\kappa, \lambda}$, we denote the simultaneous substitution of terms $\bar{t} = (t_1, \dots, t_n)$ for free variables $\bar{x} = (x_{i_1}, \dots, x_{i_n})$ with $i_j \neq i_k$ for $j \neq k$ in a term t by $t[\bar{t}/\bar{x}]$ and in a formula ϕ by $\phi[\bar{t}/\bar{x}]$. Here, we assume that quantifiers are treated by renaming the quantified variable in the sense of

$$(Qx\phi)[\bar{t}/\bar{x}] := Qz\phi[(\bar{t}', z)/(\bar{x}', x)]$$

where $Q \in \{\forall, \exists\}$, \bar{x}' is \bar{x} with x removed (if it occurs), \bar{t}' is \bar{t} with t_j removed when $x_{i_j} = x$ and z is fresh, i.e. does not occur in ϕ or \bar{t} .

Similarly to the propositional case, we consider a notion of fragments for $\mathcal{L}_{\omega_1, \omega}$ by extending the previous properties appropriately to allow for a Lindenbaum-Tarski construction over these fragments also in the first-order case later on.

Definition 3.1. A (*distributive*) *fragment* of $\mathcal{L}_{\omega_1, \omega}$ is a set $\mathcal{L}_A \subseteq \mathcal{L}_{\omega_1, \omega}$ together with a set $\text{Var}_A \subseteq \text{Var}_{\omega_1}$ and a signature $\sigma_A \subseteq \sigma$ such that \mathcal{T}_A is the set of terms of σ_A using Var_A and

- (1) $\perp \in \mathcal{L}_A$,
- (2) $P(t_1, \dots, t_n) \in \mathcal{L}_A$ for n -ary $P \in \sigma_A$ and $t_i \in \mathcal{T}_A$,
- (3) $\phi \in \mathcal{L}_A$ implies $\text{sub}(\phi) \subseteq \mathcal{L}_A$, $\text{var}(\phi) \subseteq \text{Var}_A$ and $\sigma(\phi) \subseteq \sigma_A$,
- (4) $\phi, \psi \in \mathcal{L}_A$ implies $\phi \circ \psi \in \mathcal{L}_A$ for $\circ \in \{\wedge, \vee, \rightarrow\}$ and $\exists x \phi, \forall x \phi \in \mathcal{L}_A$ for $x \in \text{Var}_A$,
- (5) $\phi \in \mathcal{L}_A$ or $t \in \mathcal{T}_A$ imply $\phi[\bar{t}/\bar{x}] \in \mathcal{L}_A$ or $t[\bar{t}/\bar{x}] \in \mathcal{T}_A$, respectively for any $\bar{t} \in (\mathcal{T}_A)^n$ and $\bar{x} = (x_{i_1}, \dots, x_{i_n}) \in (\text{Var}_A)^n$ with $i_j \neq i_k$ for $j \neq k$.
- (6) $\phi, \bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigwedge_{i \in \omega} (\phi \rightarrow \phi_i) \in \mathcal{L}_A$,
- (7) $\phi, \bigvee_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigwedge_{i \in \omega} (\phi_i \rightarrow \phi) \in \mathcal{L}_A$,
- (8) $\phi, \bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigwedge_{i \in \omega} (\phi \vee \phi_i) \in \mathcal{L}_A$.

It is additionally assumed that fragments are “saturated” when it comes to variables, in the sense that there are enough variables to find fresh ones given a finite selection of formulas from \mathcal{L}_A . More precisely, we want that for any $\phi_1, \dots, \phi_n \in \mathcal{L}_A$, there is a variable $y \in \text{Var}_A \setminus (\text{var}(\phi_1) \cup \dots \cup \text{var}(\phi_n))$.

Lemma 3.2. *For any $\Gamma \subseteq \mathcal{L}_{\omega_1, \omega}$, there is a smallest (w.r.t. \subseteq) distributive fragment $\text{frag}(\Gamma)$ such that $\Gamma \subseteq \text{frag}(\Gamma)$. If Γ is countable, then $\text{frag}(\Gamma)$ is also countable.*

Note that, for countable Γ , one can even find a countable fragment $\mathcal{L}_A \supseteq \Gamma$ with a countably infinite $Y \subseteq \text{Var}_A$ such that $\text{var}(\phi) \cap Y$ is finite for any $\phi \in \mathcal{L}_A$. So in this case, the saturation for variables is directly satisfied.

3.1. The Standard Semantics and $\mathbf{G}_{\kappa, \lambda}$. The standard semantics of infinitary first-order Gödel logics which we want to consider is, like in the propositional case, a straightforward extension of the usual finitary case. Also here, we initially focus on the full real unit interval $[0, 1]$ as the corresponding truth-value set.

Definition 3.3. An $\mathcal{L}_{\kappa, \lambda}(\sigma)$ -*Gödel-model* is a structure \mathfrak{M} which consists of

- (1) a non-empty set M ,
- (2) $P^{\mathfrak{M}} : M^n \rightarrow [0, 1]$ for every n -ary predicate P of σ ,
- (3) $f^{\mathfrak{M}} : M^n \rightarrow M$ for every n -ary function f of σ .

An $\mathcal{L}_{\kappa, \lambda}(\sigma)$ -Gödel-interpretation is a structure $\mathfrak{I} = (\mathfrak{M}, v)$ composed of an $\mathcal{L}_{\kappa, \lambda}(\sigma)$ -Gödel-model \mathfrak{M} together with a function $v : Var_{\kappa} \rightarrow M$.

Over such an interpretation \mathfrak{I} , one naturally defines the value $t^{\mathfrak{I}}$ of some term t of \mathcal{T}_{κ} . Further, we define

$$v \frac{f}{X}(x) := \begin{cases} f(x) & \text{if } x \in X, \\ v(x) & \text{otherwise,} \end{cases}$$

for $X \subseteq Var_{\kappa}$ and functions $f : X \rightarrow M$. We write $v \frac{m}{x}$ for the case of $X = \{x\}$ and $f(x) = m$ and also introduce a special notation for finite tuples with

$$v \frac{\bar{m}}{\bar{x}} := \left(\dots \left(v \frac{m_1}{x_1} \right) \dots \right) \frac{m_n}{x_n}$$

where $\bar{m} = (m_1, \dots, m_n) \in M^n$ and $\bar{x} = (x_{i_1}, \dots, x_{i_n}) \in (Var_{\kappa})^n$. We write

$$\mathfrak{I} \frac{f}{X} := (\mathfrak{M}, v \frac{f}{X})$$

and similarly for singletons and tuples. We also allow empty sets/tuples \bar{m}, \bar{x} and set $v \frac{\bar{m}}{\bar{x}} := v$ in this case.

By recursion on $\mathcal{L}_{\kappa, \lambda}$, we construct the *evaluation* $\bar{\mathfrak{I}} : \mathcal{L}_{\kappa, \lambda} \rightarrow [0, 1]$ associated with \mathfrak{I} :

- (1) $\bar{\mathfrak{I}}(\perp) := 0^{\mathbf{A}}$;
- (2) $\bar{\mathfrak{I}}(P(t_1, \dots, t_n)) := P^{\mathfrak{M}}(t_1^{\bar{\mathfrak{I}}}, \dots, t_n^{\bar{\mathfrak{I}}})$ for n -ary P ;
- (3) $\bar{\mathfrak{I}}(\phi \wedge \psi) := \min\{\bar{\mathfrak{I}}(\phi), \bar{\mathfrak{I}}(\psi)\}$;
- (4) $\bar{\mathfrak{I}}(\phi \vee \psi) := \max\{\bar{\mathfrak{I}}(\phi), \bar{\mathfrak{I}}(\psi)\}$;
- (5) $\bar{\mathfrak{I}}(\phi \rightarrow \psi) := \bar{\mathfrak{I}}(\phi) \Rightarrow \bar{\mathfrak{I}}(\psi)$;
- (6) $\bar{\mathfrak{I}}(\bigwedge \Phi) := \inf\{\bar{\mathfrak{I}}(\phi) \mid \phi \in \Phi\}$;
- (7) $\bar{\mathfrak{I}}(\bigvee \Phi) := \sup\{\bar{\mathfrak{I}}(\phi) \mid \phi \in \Phi\}$;
- (8) $\bar{\mathfrak{I}}(\forall X \phi) := \inf\{\bar{\mathfrak{I}} \frac{f}{X}(\phi) \mid f : X \rightarrow M\}$;
- (9) $\bar{\mathfrak{I}}(\exists X \phi) := \sup\{\bar{\mathfrak{I}} \frac{f}{X}(\phi) \mid f : X \rightarrow M\}$.

As before, one immediately derives a notion of semantical consequence from the model/interpretation construction and their corresponding evaluations: for $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\kappa, \lambda}$, we write $\Gamma \models_{\mathfrak{G}_{\kappa, \lambda}} \phi$ if $\bar{\mathfrak{I}}[\Gamma] \subseteq \{1\}$ implies $\bar{\mathfrak{I}}(\phi) = 1$ for any $\mathcal{L}_{\kappa, \lambda}$ -Gödel-interpretation \mathfrak{I} .

We similarly define the κ, λ -infinitary Gödel logic to be the set of consequences $\Gamma \models_{\mathfrak{G}_{\kappa, \lambda}} \phi$ and in general denote it by $\mathfrak{G}_{\kappa, \lambda}$.

Similar to the propositional case, we can also consider a different consequence relation in this first-order context which we may define by

$$\Gamma \models_{\mathfrak{G}_{\kappa, \lambda}}^{\leq} \phi \text{ if } \inf \bar{\mathfrak{I}}[\Gamma] \leq \bar{\mathfrak{I}}(\phi) \text{ for any interpretation } \mathfrak{I}.$$

Also here, $\models_{\mathfrak{G}_{\omega_1, \omega}}^{\leq}$ can be shown to coincide with $\models_{\mathfrak{G}_{\omega_1, \omega}}$ for countable sets of premises which we will show later (see Proposition 4.16).

3.2. A Proof Calculus for $\kappa = \omega_1, \lambda = \omega$. The following proof calculus $\mathcal{G}_{\omega_1, \omega}$ is obtained by extending the previous proof calculus for the propositional case with appropriate axioms and rules for the quantifiers (see again [37]).

The Calculus $\mathcal{G}_{\omega_1, \omega}$.

- (IL): the complete set of axiom schemes for propositional intuitionistic logic considered before, now in the first-order language;
- (GL): $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$;
- ($\omega\vee$): $\phi_j \rightarrow \bigvee_{i \in \omega} \phi_i$, ($j \in \omega$);
- ($\omega\wedge$): $\bigwedge_{i \in \omega} \phi_i \rightarrow \phi_j$, ($j \in \omega$);
- ($\forall E$): $\forall x \phi \rightarrow \phi[t/x]$;
- ($\exists E$): $\phi[t/x] \rightarrow \exists x \phi$;
- (MP): from $\phi \rightarrow \psi$ and ϕ , infer ψ ;
- (R ω)₁: from $\phi_i \rightarrow \psi$ for $i \in \omega$, infer $\bigvee_{i \in \omega} \phi_i \rightarrow \psi$;
- (R ω)₂: from $\phi \rightarrow \psi_i$ for $i \in \omega$, infer $\phi \rightarrow \bigwedge_{i \in \omega} \psi_i$;
- ($\forall I$): from $\psi \rightarrow \phi$, infer $\psi \rightarrow \forall x \phi$ where $x \notin \text{free}(\psi)$;

($\exists I$): from $\phi \rightarrow \psi$, infer $\exists x\phi \rightarrow \psi$ where $x \notin \text{free}(\psi)$.

As before, we consider an extension $\mathcal{G}_{\omega_1, \omega}^D$ obtained by adding the scheme

$$(D) \quad \bigwedge_{i \in \omega} (\phi \vee \psi_i) \rightarrow \left(\phi \vee \bigwedge_{i \in \omega} \psi_i \right)$$

and now *additionally* also the axiom scheme

$$(QS) \quad \forall x(\psi \vee \phi) \rightarrow (\psi \vee \forall x\phi) \text{ where } x \notin \text{free}(\psi).$$

The notion of proof immediately transfers to this setting from the propositional case. We write $\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}} \phi$ (or $\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^D} \phi$, respectively) if there is such a proof. We define restrictions $\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_A)$ (or $\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_A)$) to some fragment \mathcal{L}_A of $\mathcal{L}_{\omega_1, \omega}$ as before.

Note that also both $\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_A)$ and $\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_A)$ have the classical Deduction Theorem.

Lemma 3.4. *For any $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_A$ and ϕ closed, we have*

$$\Gamma \cup \{\phi\} \vdash_{\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_A)} \psi \text{ iff } \Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_A)} \phi \rightarrow \psi.$$

The same holds for $\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_A)$.

4. L-ALGEBRAS, CHAINS AND ALGEBRAIC SEMANTICS

We follow a similar route to semantic completeness as in the setting of finitary Gödel logics (see in particular [8, 21]): we first establish completeness w.r.t. a class of algebras and then construct embeddings from that class into the relevant structures of the intended interpretation.

More precisely, we first show completeness w.r.t. linearly ordered and sufficiently complete Heyting algebras over countable fragments and then extend this to the Heyting algebra of the real unit interval by embeddings, similar to [21]. This approach does not only offer a high degree of modularity but also establishes linear Heyting algebras (with sufficient completeness) as the algebraic semantics for infinitary Gödel logics, in analogy to the finitary case. Once we have established the result with respect to countable fragments, this assumption can be removed over complete algebras like the unit interval.

For that, we need various notions from the theory of Heyting algebras and the next subsection gives a, for reasons of self-containedness, quite detailed account mostly following [21, 29] (up to some notation change).

4.1. Heyting algebras and related notions. A *Heyting algebra* is a structure $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ such that $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ is a bounded lattice with largest element $1^{\mathbf{A}}$ and smallest element $0^{\mathbf{A}}$ and $\rightarrow^{\mathbf{A}}$ is a binary operation with

- (1) $x \rightarrow^{\mathbf{A}} x = 1^{\mathbf{A}}$,
- (2) $x \wedge^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} y) = x \wedge^{\mathbf{A}} y$,
- (3) $y \wedge^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} y) = y$,
- (4) $x \rightarrow^{\mathbf{A}} (y \wedge^{\mathbf{A}} z) = (x \rightarrow^{\mathbf{A}} y) \wedge^{\mathbf{A}} (x \rightarrow^{\mathbf{A}} z)$,

where we write $a \leq^{\mathbf{A}} b$ for $a \wedge^{\mathbf{A}} b = a$ and $\neg^{\mathbf{A}} x := x \rightarrow^{\mathbf{A}} 0^{\mathbf{A}}$. *Joins (suprema)* and *meets (infima)* of subsets X are defined as usual and denoted $\bigvee^{\mathbf{A}} X$ and $\bigwedge^{\mathbf{A}} X$, respectively. If every subset has a join and meet, \mathbf{A} is called *complete*. An existing meet $\bigwedge^{\mathbf{A}} X$ is called *distributive* if

$$\bigwedge_{x \in X}^{\mathbf{A}} (y \vee^{\mathbf{A}} x) = y \vee^{\mathbf{A}} \bigwedge^{\mathbf{A}} X$$

for any $y \in \mathbf{A}$ and \mathbf{A} is called *distributive* if every meet is distributive.

Two particular types of Heyting algebras, which we will consider in this note are *chains*, i.e. Heyting algebras where $\leq^{\mathbf{A}}$ is linear, and *L-algebras*, i.e. Heyting algebras where $(x \rightarrow^{\mathbf{A}} y) \vee^{\mathbf{A}} (y \rightarrow^{\mathbf{A}} x) = 1^{\mathbf{A}}$ for all $x, y \in \mathbf{A}$. We denote the class of all L-algebras by \mathbf{L} , the class of all distributive L-algebras by \mathbf{DL} and the class of all chains by \mathbf{C} . We write \mathbf{CC} for the class of countable chains. Naturally, every chain is a distributive L-algebra.

Further, we will need the notion of a filter. A set $F \subseteq \mathbf{A}$ is a *filter* for a Heyting algebra \mathbf{A} , if (1) $1 \in F$, (2) $x, y \in F$ implies $x \wedge^{\mathbf{A}} y \in F$ and (3) $x \in F$ and $x \leq^{\mathbf{A}} y$ imply $y \in F$. F is called *proper* if $F \subsetneq \mathbf{A}$ and F is called a *prime filter* if it is proper and if $x \vee^{\mathbf{A}} y \in F$ implies $x \in F$ or $y \in F$.

We then can “filter” a Heyting algebra \mathbf{A} via F : define $x \leq_F y$ if $x \rightarrow^{\mathbf{A}} y \in F$ for $x, y \in \mathbf{A}$ and $x \equiv_F y$ if $x \leq_F y$ and $y \leq_F x$. Then \equiv_F is a congruence relation on Heyting algebras and thus defines a quotient Heyting

algebra \mathbf{A}/F over the set of equivalence classes $[a]_F$ of elements a of \mathbf{A} over \equiv_F . In particular, “filtering” with a prime filter in L-algebras yields a chain:

Lemma 4.1 ([21], Lemma 1.1). *If F is a prime filter of some L-algebra \mathbf{A} , then \mathbf{A}/F is a chain.*

The behavior of meets and joins under quotients will be of particular importance later on. For that, we first note the following:

Lemma 4.2 ([29], Chapter IV, 7.2, (7) and (8)). *Let \mathbf{A} be a Heyting algebra. If*

$$\bigwedge^{\mathbf{A}} X \text{ and } \bigvee^{\mathbf{A}} Y$$

exist in \mathbf{A} , then

$$\bigwedge_{x \in X}^{\mathbf{A}} (z \rightarrow^{\mathbf{A}} x) = z \rightarrow^{\mathbf{A}} \bigwedge^{\mathbf{A}} X \text{ and } \bigwedge_{y \in Y}^{\mathbf{A}} (y \rightarrow^{\mathbf{A}} z) = \bigvee^{\mathbf{A}} Y \rightarrow^{\mathbf{A}} z$$

for any $z \in \mathbf{A}$.

A filter F of a Heyting algebra \mathbf{A} is said to *preserve an existing meet*

$$\bar{x} = \bigwedge^{\mathbf{A}} X$$

if $\bar{x} \in F$ if, and only if $x \in F$ for all $x \in X$.

Further, a homomorphism $h : \mathbf{A} \rightarrow \mathbf{B}$ of Heyting algebras is said to *preserve a meet* $\bigwedge^{\mathbf{A}} X$, *or a join* $\bigvee^{\mathbf{A}} Y$, if

$$h \left(\bigwedge^{\mathbf{A}} X \right) = \bigwedge^{\mathbf{B}} h[X] \text{ or } h \left(\bigvee^{\mathbf{A}} Y \right) = \bigvee^{\mathbf{B}} h[Y],$$

respectively.

Lemma 4.3 ([21], Lemma 2.2). *Suppose*

$$\bigwedge^{\mathbf{A}} X \text{ and } \bigvee^{\mathbf{A}} Y$$

exist in \mathbf{A} and F is a filter of \mathbf{A} which preserves

$$\bigwedge_{x \in X}^{\mathbf{A}} (z \rightarrow^{\mathbf{A}} x) \text{ and } \bigwedge_{y \in Y}^{\mathbf{A}} (y \rightarrow^{\mathbf{A}} z)$$

for any $z \in \mathbf{A}$. Then

$$\left[\bigwedge^{\mathbf{A}} X \right]_F = \bigwedge_{x \in X}^{\mathbf{A}/F} [x]_F \text{ and } \left[\bigvee^{\mathbf{A}} Y \right]_F = \bigvee_{y \in Y}^{\mathbf{A}/F} [y]_F.$$

Therefore, the canonical map $x \mapsto [x]_F$ from \mathbf{A} into \mathbf{A}/F is a Heyting algebra homomorphism which preserves the respective meet and join.

Lemma 4.4 ([21], Theorem 2.3). *Let \mathbf{A} be a Heyting algebra and let*

$$x_n = \bigwedge^{\mathbf{A}} X_n$$

be a sequence of distributive meets in \mathbf{A} . If $x, y \in \mathbf{A}$ with $x \not\leq^{\mathbf{A}} y$, then there is a prime filter F such that $x \in F$, $y \notin F$ and such that F preserves the meets given by x_n .

By $[0, 1]_{\mathbb{Q}}$ and $[0, 1]_{\mathbb{R}}$, we denote the Heyting algebras of all rationals in the unit interval and of the whole unit interval, respectively.

Lemma 4.5 ([21], Lemma 3.7). *Let \mathbf{A} be a countable chain. Then there is an embedding, i.e. an injective homomorphism of Heyting algebras*

$$q : \mathbf{A} \rightarrow [0, 1]_{\mathbb{Q}}$$

which preserves all meets and joins of \mathbf{A} .

4.2. Algebraic Propositional Evaluations for \mathcal{L}_{ω_1} . In this section, we now introduce the actual algebraic generalizations of the \mathcal{L}_{ω_1} -Gödel-evaluations, broadening the domains to fragments and the range to certain Heyting algebras which may be, in a particular way, incomplete. This will be necessary in the approach to completeness chosen here since the Lindenbaum-Tarski algebras later constructed are, in fact, incomplete.

Let \mathcal{L}_A be an arbitrary fragment of \mathcal{L}_{ω_1} and \mathbf{A} be a Heyting algebra.

Definition 4.6. A function $v : \mathcal{L}_A \rightarrow \mathbf{A}$ is an (\mathbf{A} -valued) \mathcal{L}_A -evaluation if

- (1) $v(\perp) = 0^{\mathbf{A}}$,
- (2) $v(\phi \circ \psi) = v(\phi) \circ^{\mathbf{A}} v(\psi)$ for $\circ \in \{\rightarrow, \wedge, \vee\}$,
- (3) for any $\bigwedge_{i \in \omega} \phi_i, \bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$, the meet and join of $\{v(\phi_i) \mid i \in \omega\}$ exist and we have

$$v\left(\bigwedge_{i \in \omega} \phi_i\right) = \bigwedge_{i \in \omega}^{\mathbf{A}} v(\phi_i)$$

and

$$v\left(\bigvee_{i \in \omega} \phi_i\right) = \bigvee_{i \in \omega}^{\mathbf{A}} v(\phi_i).$$

Given such an evaluation v , we still write $v[\Gamma] := \{v(\gamma) \mid \gamma \in \Gamma\}$ for sets $\Gamma \subseteq \mathcal{L}_A$ and $(\mathbf{A}, v) \models \phi$ for $v(\phi) = 1^{\mathbf{A}}$. We denote the set of all \mathbf{A} -valued \mathcal{L}_A -evaluations by $\text{Ev}(\mathcal{L}_A; \mathbf{A})$.

Using this notion of \mathcal{L}_A -evaluations, there is now a natural notion of semantical entailment: let Cl be a class of Heyting algebras and let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$. We write $\Gamma \models_{\text{Cl}(\mathcal{L}_A)} \phi$ if

$$\forall \mathbf{A} \in \text{Cl} \forall v \in \text{Ev}(\mathcal{L}_A; \mathbf{A}) (v[\Gamma] \subseteq \{1^{\mathbf{A}}\} \text{ implies } v(\phi) = 1^{\mathbf{A}}).$$

We abbreviate $\Gamma \models_{\text{Cl}(\mathcal{L}_{\omega_1})} \phi$ by $\Gamma \models_{\text{Cl}} \phi$.

By transfinite induction on the length of the proof, one quickly verifies the following soundness result:

Lemma 4.7. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$:

- (1) $\Gamma \vdash_{\mathcal{G}_{\omega_1}(\mathcal{L}_A)} \phi$ implies $\Gamma \models_{\text{L}(\mathcal{L}_A)} \phi$,
- (2) $\Gamma \vdash_{\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)} \phi$ implies $\Gamma \models_{\text{DL}(\mathcal{L}_A)} \phi$.

To form new \mathcal{L}_A -evaluations by composition with homomorphisms of Heyting algebras, we now have to additionally require that the existing meets and joins arising from the infinitary connectives are preserved. This is captured in the following lemma.

Lemma 4.8. Let $v : \mathcal{L}_A \rightarrow \mathbf{A}$ be an evaluation of \mathcal{L}_A in \mathbf{A} and let $h : \mathbf{A} \rightarrow \mathbf{B}$ be a Heyting algebra homomorphism which preserves all the meets and joins

$$\bigwedge_{i \in \omega}^{\mathbf{A}} v(\phi_i) \text{ and } \bigvee_{i \in \omega}^{\mathbf{A}} v(\psi_i)$$

for any $\bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ and $\bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$. Then $h \circ v$ is an evaluation of \mathcal{L}_A in \mathbf{B} .

The proof is rather immediate and thus omitted.

The following result is a natural generalization of a finitary result (for first-order variants, however) given in [8] (see Lemma 2.14) and the proof is essentially the same (and thus omitted).

Lemma 4.9. Let \mathbf{A} be a complete linear Heyting algebra, $a \in \mathbf{A}$, and let $v : \mathcal{L}_{\kappa} \rightarrow \mathbf{A}$ be any evaluation. Then, for

$$v_a(x) := \begin{cases} v(x) & \text{if } v(x) <^{\mathbf{A}} a, \\ 1^{\mathbf{A}} & \text{otherwise,} \end{cases}$$

with $x \in \text{Var}_{\kappa}$, the unique extension $\overline{v}_a : \mathcal{L}_{\kappa} \rightarrow \mathbf{A}$ to an evaluation satisfies:

$$a \notin v[\text{sub}(\phi)] \text{ implies } \overline{v}_a(\phi) = \begin{cases} v(\phi) & \text{if } v(\phi) <^{\mathbf{A}} a, \\ 1^{\mathbf{A}} & \text{otherwise,} \end{cases}$$

for any $\phi \in \mathcal{L}_{\kappa}$.

As a direct consequence, we obtain the following result (similar to Proposition 2.15 in [8]) from this:

Proposition 4.10. For any countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\omega_1}$, $\Gamma \models_{\mathcal{G}_{\omega_1}}^{\leq} \phi$ iff $\Gamma \models_{\mathcal{G}_{\omega_1}} \phi$.

4.3. Algebraic First-Order Interpretations for $\mathcal{L}_{\omega_1, \omega}$. For a similar motivation as in the propositional case, we are also lead to broadening the definitions of models and interpretations for $\mathcal{L}_{\omega_1, \omega}$ to both arbitrary fragments as domains and (sufficiently complete) Heyting algebras as ranges. For this, let \mathcal{L}_A now be a fragment of $\mathcal{L}_{\omega_1, \omega}$ and \mathbf{A} again be a Heyting algebra.

Definition 4.11. An \mathcal{L}_A -model (over \mathbf{A}) is a structure \mathfrak{M} which consists of

- (1) a Heyting algebra \mathbf{A} ,
- (2) a non-empty set M ,
- (3) $P^{\mathfrak{M}} : M^n \rightarrow \mathbf{A}$ for every n -ary predicate P of σ_A ,
- (4) $f^{\mathfrak{M}} : M^n \rightarrow M$ for every n -ary function f of σ_A .

An \mathcal{L}_A -interpretation (over \mathbf{A}) is a structure $\mathfrak{J} = (\mathfrak{M}, v)$ where \mathfrak{M} is an \mathcal{L}_A -model and $v : Var_A \rightarrow M$.

As before, over some \mathcal{L}_A -interpretation \mathfrak{J} , one naturally defines the value of some term t of \mathcal{T}_A which we still denote by $t^{\mathfrak{J}}$. $v^{\frac{m}{x}}$, $v^{\frac{\bar{m}}{x}}$ and the resulting $\bar{\mathfrak{J}}^{\frac{m}{x}}$ are defined (over Var_A) in the same way as with the standard semantics.

Such a model \mathfrak{M} is called (\mathcal{L}_A -)suitable for a variable assignment $v : Var_A \rightarrow M$ if for any $\bar{m} \in M^n$ and $\bar{x} \in (Var_A)^n$, corresponding to the interpretation $\mathfrak{J} = (\mathfrak{M}, v^{\frac{\bar{m}}{\bar{x}}})$, there is a function $\bar{\mathfrak{J}} : \mathcal{L}_A \rightarrow \mathbf{A}$ such that

- (1) $\bar{\mathfrak{J}}(\perp) = 0^{\mathbf{A}}$,
- (2) $\bar{\mathfrak{J}}(P(t_1, \dots, t_n)) = P^{\mathfrak{M}}(t_1^{\bar{\mathfrak{J}}}, \dots, t_n^{\bar{\mathfrak{J}}})$ for n -ary $P \in \sigma_A$,
- (3) $\bar{\mathfrak{J}}(\phi \circ \psi) = \bar{\mathfrak{J}}(\phi) \circ^{\mathbf{A}} \bar{\mathfrak{J}}(\psi)$ for $\circ \in \{\wedge, \vee, \rightarrow\}$,
- (4) for any $\bigwedge_{i \in \omega} \phi_i, \bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$, the meet and join of $\{\bar{\mathfrak{J}}(\phi_i) \mid i \in \omega\}$ exist and we have

$$\bar{\mathfrak{J}}\left(\bigwedge_{i \in \omega} \phi_i\right) = \bigwedge_{i \in \omega}^{\mathbf{A}} \bar{\mathfrak{J}}(\phi_i)$$

as well as

$$\bar{\mathfrak{J}}\left(\bigvee_{i \in \omega} \phi_i\right) = \bigvee_{i \in \omega}^{\mathbf{A}} \bar{\mathfrak{J}}(\phi_i),$$

- (5) for any $\phi \in \mathcal{L}_A$ and any $x \in Var_A$, the meet and join of $\{\bar{\mathfrak{J}}^{\frac{m}{x}}(\phi) \mid m \in M\}$ exist and we have

$$\bar{\mathfrak{J}}(\forall x \phi) = \bigwedge_{m \in M}^{\mathbf{A}} \bar{\mathfrak{J}}^{\frac{m}{x}}(\phi)$$

as well as

$$\bar{\mathfrak{J}}(\exists x \phi) = \bigvee_{m \in M}^{\mathbf{A}} \bar{\mathfrak{J}}^{\frac{m}{x}}(\phi).$$

Of course, the existence of a suitable model for a variable assignment is not guaranteed if the variable assignment is such that the necessary meets or joins do not exist. However, if a model \mathfrak{M} is suitable for a variable assignment v , then $\bar{\mathfrak{J}}$ is unique for all $\mathfrak{J} = (\mathfrak{M}, v^{\frac{\bar{m}}{\bar{x}}})$.

Remark 4.12. Note that it actually suffices to establish the existence of such an extension only for \bar{m} and $\bar{x} = (x_{i_1}, \dots, x_{i_n})$ where $i_j \neq i_k$ for $j \neq k$.

We still write $\bar{\mathfrak{J}}[\Gamma] := \{\bar{\mathfrak{J}}(\gamma) \mid \gamma \in \Gamma\}$ for sets $\Gamma \subseteq \mathcal{L}_A$ and $\bar{\mathfrak{J}} \models \phi$ if $\bar{\mathfrak{J}}(\phi) = 1^{\mathbf{A}}$.

We denote the class of all \mathcal{L}_A -models over \mathbf{A} by $\text{Mod}(\mathcal{L}_A; \mathbf{A})$ and if \mathfrak{M} is a model, we write $\text{Int}(\mathcal{L}_A; \mathfrak{M})$ for the set of all corresponding interpretations (\mathfrak{M}, v) such that \mathfrak{M} is suitable for v .

The derived notion of semantic consequence is then given by the following: let Cl be a class of Heyting algebras and let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$. We write $\Gamma \models_{\text{Cl}(\mathcal{L}_A)} \phi$ if

$$\forall \mathbf{A} \in \text{Cl} \forall \mathfrak{M} \in \text{Mod}(\mathcal{L}_A; \mathbf{A}) \forall \mathfrak{J} \in \text{Int}(\mathcal{L}_A; \mathfrak{M}) (\bar{\mathfrak{J}}[\Gamma] \subseteq \{1^{\mathbf{A}}\} \text{ implies } \bar{\mathfrak{J}}(\phi) = 1^{\mathbf{A}}).$$

It is again straightforward to verify the following soundness result.

Lemma 4.13. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$ where all formulas from Γ are closed:

- (1) $\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_A)} \phi$ implies $\Gamma \models_{\text{L}(\mathcal{L}_A)} \phi$,
- (2) $\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_A)} \phi$ implies $\Gamma \models_{\text{DL}(\mathcal{L}_A)} \phi$.

As before, we can form new interpretations by composition with Heyting algebra homomorphisms as long as they respect the necessary meets and joins.

Lemma 4.14. *Let \mathfrak{M} be a model over some Heyting algebra \mathbf{A} which is suitable for $v : \text{Var}_{\mathbf{A}} \rightarrow M$ and let $h : \mathbf{A} \rightarrow \mathbf{B}$ be a Heyting algebra homomorphism which, for any interpretation $\mathfrak{J} = (\mathfrak{M}, v \frac{\overline{}}{x})$ over \mathfrak{M} , preserves all the meets and joins*

$$\bigwedge_{i \in \omega} \overline{\mathfrak{J}}(\phi_i) \text{ and } \bigvee_{i \in \omega} \overline{\mathfrak{J}}(\psi_i)$$

for any $\bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_{\mathbf{A}}$ and $\bigvee_{i \in \omega} \psi_i \in \mathcal{L}_{\mathbf{A}}$ as well as

$$\bigwedge_{m \in M} \overline{\mathfrak{J}} \frac{\overline{m}}{x}(\phi) \text{ and } \bigvee_{m \in M} \overline{\mathfrak{J}} \frac{\overline{m}}{x}(\phi)$$

for all $\phi \in \mathcal{L}_{\mathbf{A}}$, $x \in \text{Var}_{\mathbf{A}}$ and $m \in M$. Then the model $h \circ \mathfrak{M}$ defined over the same domain with $f^{h \circ \mathfrak{M}} := f^{\mathfrak{M}}$ for any function symbol f and $P^{h \circ \mathfrak{M}} := h \circ P^{\mathfrak{M}}$ for predicate symbols P is a suitable model for v and for any interpretation $\mathfrak{J} = (\mathfrak{M}, v \frac{\overline{}}{x})$, we have

$$\overline{h \circ \mathfrak{J}}(\phi) = h(\overline{\mathfrak{J}}(\phi))$$

for all $\phi \in \mathcal{L}_{\mathbf{A}}$ where $h \circ \mathfrak{J} = (h \circ \mathfrak{M}, v \frac{\overline{}}{x})$.

We then obtain a result analogous to Lemma 4.9. This first-order version is rather intricate to formulate if one wants to guarantee a similar level of generality as in the finitary case. We give it here in its full strength but will later mostly use the special case with $\kappa = \omega_1$ and $\lambda = \omega$ where the conditions simplify considerably (in particular the definition of $\text{Val}_{\mathfrak{J}}(\phi)$).

Lemma 4.15. *Let \mathbf{A} be a complete linear Heyting algebra, $a \in A$, and let \mathfrak{M} be an \mathbf{A} -valued model. Define \mathfrak{M}_a from \mathfrak{M} by replacing $P^{\mathfrak{M}}$ with*

$$P^{\mathfrak{M}_a}(m_1, \dots, m_n) := \begin{cases} P^{\mathfrak{M}}(m_1, \dots, m_n) & \text{if } P^{\mathfrak{M}}(m_1, \dots, m_n) <^{\mathbf{A}} a, \\ 1^{\mathbf{A}} & \text{otherwise,} \end{cases}$$

where P is an n -ary predicate. We write $\mathfrak{J}_a = (\mathfrak{M}_a, w)$ and

$$\text{Val}_{\mathfrak{J}}(\phi) := \left\{ \overline{\mathfrak{J}}(\psi) \mid \psi \in \text{sub}(\phi) \text{ and } \mathfrak{J}' = \left(\dots \left(\mathfrak{J} \frac{f_1}{X_1} \right) \frac{f_2}{X_2} \dots \right) \frac{f_n}{X_n} \right. \\ \left. \text{where } X_i \subseteq \text{Var}_{\kappa} \cap \text{sub}(\phi) \text{ with } |X_i| < \lambda \text{ and } f_i : X_i \rightarrow M \right\}$$

given an interpretation $\mathfrak{J} = (\mathfrak{M}, w)$.

Then, for a given $v : \text{Var}_{\kappa} \rightarrow M$ and $\mathfrak{J} = (\mathfrak{M}, v)$, we have

$$a \notin \text{Val}_{\mathfrak{J}'}(\phi) \text{ implies } \overline{\mathfrak{J}}'_a(\phi) = \begin{cases} \overline{\mathfrak{J}}'(\phi) & \text{if } \overline{\mathfrak{J}}'(\phi) <^{\mathbf{A}} a, \\ 1^{\mathbf{A}} & \text{otherwise,} \end{cases}$$

for any $\phi \in \mathcal{L}_{\kappa, \lambda}$ and any interpretation

$$\mathfrak{J}' = \left(\dots \left(\mathfrak{J} \frac{f_1}{X_1} \right) \frac{f_2}{X_2} \dots \right) \frac{f_n}{X_n}$$

where $X_i \subseteq \text{Var}_{\kappa}$ with $|X_i| < \lambda$ and $f_i : X_i \rightarrow M$.

As before, this immediately yields:

Proposition 4.16. *Similarly, for any countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\omega_1, \omega}$, we have $\Gamma \models_{\mathfrak{G}_{\omega_1, \omega}}^{\leq} \phi$ iff $\Gamma \models_{\mathfrak{G}_{\omega_1, \omega}} \phi$.*

5. A PROPOSITIONAL AND FIRST-ORDER COMPLETENESS THEOREM

We fix a fragment $\mathcal{L}_{\mathbf{A}}$ of either \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$ and all notions, if not explicitly indicated, are to be understood relative to that fragment. Let $\mathcal{G}^{(D)}(\mathcal{L}_{\mathbf{A}})$ be either $\mathcal{G}_{\omega_1}(\mathcal{L}_{\mathbf{A}})$ or $\mathcal{G}_{\omega_1}^D(\mathcal{L}_{\mathbf{A}})$ in the propositional case or either $\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_{\mathbf{A}})$ or $\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_{\mathbf{A}})$ in the first order case.

We construct, as it is usually done, the Lindenbaum-Tarski algebra of $\mathcal{G}^{(D)}(\mathcal{L}_{\mathbf{A}})$: given $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_{\mathbf{A}}$, we write $\phi \preceq^{\Gamma} \psi$ if

$$\Gamma \vdash_{\mathcal{G}^{(D)}(\mathcal{L}_{\mathbf{A}})} \phi \rightarrow \psi$$

and write $\phi \equiv^{\Gamma} \psi$ if

$$\phi \preceq^{\Gamma} \psi \text{ and } \psi \preceq^{\Gamma} \phi.$$

We write $[\phi]^{\Gamma}$ for the equivalence class of ϕ under \equiv^{Γ} and $\mathcal{L}_{\mathbf{A}} / \equiv^{\Gamma}$ for the set of all equivalence classes. In the following, we even omit the Γ when the context is clear. The *Lindenbaum-Tarski algebra* \mathbf{LT}^{Γ} is defined as

$$\mathbf{LT}^{\Gamma} := \langle \mathcal{L}_{\mathbf{A}} / \equiv^{\Gamma}, \wedge^{\mathbf{LT}}, \vee^{\mathbf{LT}}, \rightarrow^{\mathbf{LT}}, 0^{\mathbf{LT}}, 1^{\mathbf{LT}} \rangle$$

where we define

- (1) $[\phi] \wedge^{\mathbf{LT}} [\psi] := [\phi \wedge \psi]$,
- (2) $[\phi] \vee^{\mathbf{LT}} [\psi] := [\phi \vee \psi]$,
- (3) $[\phi] \rightarrow^{\mathbf{LT}} [\psi] := [\phi \rightarrow \psi]$,
- (4) $0^{\mathbf{LT}} := [\perp]$,
- (5) $1^{\mathbf{LT}} := [\top]$.

Further, the order induced on \mathbf{LT}^Γ is given by $[\phi] \leq^{\mathbf{LT}} [\psi]$ iff $\phi \preceq^\Gamma \psi$

Lemma 5.1. *We have:*

- (1) \mathbf{LT}^Γ is a well-defined L -algebra;
- (2)

$$\bigwedge_{i \in \omega}^{\mathbf{LT}} [\phi_i] = \left[\bigwedge_{i \in \omega} \phi_i \right] \quad \text{and} \quad \bigvee_{i \in \omega}^{\mathbf{LT}} [\psi_i] = \left[\bigvee_{i \in \omega} \psi_i \right]$$

for $\bigwedge_{i \in \omega} \phi_i, \bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$;

- (3) In the case of $\mathcal{G}^D(\mathcal{L}_A)$, the meets

$$\bigwedge_{i \in \omega}^{\mathbf{LT}} ([\chi] \rightarrow^{\mathbf{LT}} [\phi_i]) \quad \text{and} \quad \bigwedge_{i \in \omega}^{\mathbf{LT}} ([\psi_i] \rightarrow^{\mathbf{LT}} [\chi])$$

are distributive for every additional $\chi \in \mathcal{L}_A$.

- (4)

$$[\forall x \phi] = \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[t/x]] \quad \text{and} \quad [\exists x \phi] = \bigvee_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[t/x]]$$

for $\phi \in \mathcal{L}_A$ and $x \in \text{Var}_A$ in the first-order case;

- (5) In the case of $\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_A)$, the meets

$$\bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} ([\chi] \rightarrow^{\mathbf{LT}} [\phi[t/x]]) \quad \text{and} \quad \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} ([\phi[t/x]] \rightarrow^{\mathbf{LT}} [\chi])$$

are distributive for every additional $\chi \in \mathcal{L}_A$.

Proof. (1) We skip the finitary operations. Using the axiom scheme (GL), it is easy to see that \mathbf{LT}^Γ is a well-defined L -algebra.

- (2) Let $\bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ and $\bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$. Then, we have

$$\mathcal{G}^{(D)}(\mathcal{L}_A) \vdash \bigwedge_{i \in \omega} \phi_i \rightarrow \phi_j \quad \text{and} \quad \mathcal{G}^{(D)}(\mathcal{L}_A) \vdash \psi_j \rightarrow \bigvee_{i \in \omega} \psi_i$$

for any j by axioms $(\omega \wedge)$ and $(\omega \vee)$ which gives

$$\left[\bigwedge_{i \in \omega} \phi_i \right] \leq^{\mathbf{LT}} [\phi_j] \quad \text{and} \quad [\psi_j] \leq^{\mathbf{LT}} \left[\bigvee_{i \in \omega} \psi_i \right]$$

for any j . Suppose that $[\chi] \leq^{\mathbf{LT}} [\phi_j]$ for any j and $[\psi_j] \leq^{\mathbf{LT}} [\chi']$ for any j . Therefore, we have

$$\Gamma \vdash_{\mathcal{G}^{(D)}(\mathcal{L}_A)} \chi \rightarrow \phi_j \quad \text{and} \quad \Gamma \vdash_{\mathcal{G}^{(D)}(\mathcal{L}_A)} \psi_j \rightarrow \chi'$$

for any j which implies

$$\Gamma \vdash_{\mathcal{G}^{(D)}(\mathcal{L}_A)} \chi \rightarrow \bigwedge_{i \in \omega} \phi_i \quad \text{and} \quad \Gamma \vdash_{\mathcal{G}^{(D)}(\mathcal{L}_A)} \bigvee_{i \in \omega} \psi_i \rightarrow \chi'$$

by $(R\omega)_{1,2}$ which is

$$[\chi] \leq^{\mathbf{LT}} \left[\bigwedge_{i \in \omega} \phi_i \right] \quad \text{and} \quad \left[\bigvee_{i \in \omega} \psi_i \right] \leq^{\mathbf{LT}} [\chi'].$$

This gives that

$$\bigwedge_{i \in \omega}^{\mathbf{LT}} [\phi_i] = \left[\bigwedge_{i \in \omega} \phi_i \right] \quad \text{and} \quad \bigvee_{i \in \omega}^{\mathbf{LT}} [\psi_i] = \left[\bigvee_{i \in \omega} \psi_i \right].$$

- (3) Let $\chi, \xi \in \mathcal{L}_A$ be arbitrary. We have

$$\bigwedge_{i \in \omega} (\xi \vee (\psi_i \rightarrow \chi)) \in \mathcal{L}_A \quad \text{and} \quad \bigwedge_{i \in \omega} (\xi \vee (\chi \rightarrow \phi_i)) \in \mathcal{L}_A.$$

by the closure properties of \mathcal{L}_A . Write α_i for either $\chi \rightarrow \phi_i$ or $\psi_i \rightarrow \chi$. Now, we get

$$[\xi] \vee^{\mathbf{LT}} \bigwedge_{i \in \omega}^{\mathbf{LT}} [\alpha_i] \leq^{\mathbf{LT}} [\xi] \vee^{\mathbf{LT}} [\alpha_j]$$

for any j by axiom $(\omega \wedge)$ and therefore

$$[\xi] \vee^{\mathbf{LT}} \bigwedge_{i \in \omega}^{\mathbf{LT}} [\alpha_i] \leq^{\mathbf{LT}} \bigwedge_{i \in \omega}^{\mathbf{LT}} ([\xi] \vee^{\mathbf{LT}} [\alpha_i]) = \left[\bigwedge_{i \in \omega} (\xi \vee \alpha_i) \right]$$

using item (2). Further, by axiom scheme (D) , we have

$$\Gamma \vdash_{\mathcal{G}^D(\mathcal{L}_A)} \bigwedge_{i \in \omega} (\xi \vee \alpha_i) \rightarrow \left(\xi \vee \bigwedge_{i \in \omega} \alpha_i \right)$$

which gives the converse

$$\left[\bigwedge_{i \in \omega} (\xi \vee \alpha_i) \right] \leq^{\mathbf{LT}} [\xi] \vee^{\mathbf{LT}} \bigwedge_{i \in \omega}^{\mathbf{LT}} [\alpha_i],$$

i.e. combined we have

$$[\xi] \vee^{\mathbf{LT}} \bigwedge_{i \in \omega}^{\mathbf{LT}} [\alpha_i] = \left[\bigwedge_{i \in \omega} (\xi \vee \alpha_i) \right].$$

- (4) Both quantifier claims can be proved as outlined in [21]. We still sketch the proofs here for self-containedness.

For the first two quantifier claims, note that we have

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \forall x \phi \rightarrow \phi[t/x] \text{ and } \Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \phi[t/x] \rightarrow \exists x \phi$$

for any $t \in \mathcal{T}_A$ by the axioms $(\forall E)$ and $(\exists E)$. Now, suppose that $\chi \in \mathcal{L}_A$ is such that

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \chi \rightarrow \phi[t/x] \text{ for all } t \in \mathcal{T}_A \text{ or } \Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \phi[t/x] \rightarrow \chi \text{ for all } t \in \mathcal{T}_A.$$

Then, pick $y \in \text{Var}_A$ with $y \notin \text{var}(\chi) \cup \text{var}(\phi)$. Note that this is possible as fragments are saturated for variables. By assumption, we have

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \chi \rightarrow \phi[y/x] \text{ or } \Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \phi[y/x] \rightarrow \chi$$

and as y is not free in χ , we get

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \chi \rightarrow \forall y \phi[y/x] \text{ or } \Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \exists y \phi[y/x] \rightarrow \chi$$

via the rule $(\forall I)$ and $(\exists I)$. As we have

$$\vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \forall y \phi[y/x] \leftrightarrow \forall x \phi \text{ and } \vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \exists y \phi[y/x] \leftrightarrow \exists x \phi,$$

the claims follow.

- (5) Let $\phi \in \mathcal{L}_A$ and $x \in \text{Var}_A$ as well as $\chi, \xi \in \mathcal{L}_A$. Further, let $y \in \text{Var}_A$ with $y \notin \text{var}(\phi) \cup \text{var}(\chi) \cup \text{var}(\xi)$ (note again variable saturation). Then, we have

$$\vdash_{\mathcal{G}_{\omega_1, \omega}^{(D)}(\mathcal{L}_A)} \phi[t/x] \leftrightarrow \phi[y/x][t/y]$$

and therefore, defining $\phi' := \phi[y/x]$, we have $[\phi[t/x]] = [\phi'[t/y]]$. Since y does not occur in χ , we have

$$\chi \rightarrow \phi'[t/y] = (\chi \rightarrow \phi')[t/y] =: \alpha^\forall[t/y]$$

and

$$\phi'[t/y] \rightarrow \chi = (\phi' \rightarrow \chi)[t/y] =: \alpha^\exists[t/y]$$

and therefore, since y also does not occur in ξ , the previous quantifier claims imply (where we write α for either α^\exists or α^\forall):

$$\begin{aligned} \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} ([\xi] \vee^{\mathbf{LT}} [\alpha[t/y]]) &= \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [(\xi \vee \alpha)[t/y]] \\ &= [\forall y (\xi \vee \alpha)]. \end{aligned}$$

Here, we have used item (4). Using the distributivity axiom, as y again does not occur in ξ , we get

$$\begin{aligned} [\forall y (\xi \vee \alpha)] &= [\xi \vee \forall y \alpha] \\ &= [\xi] \vee^{\mathbf{LT}} \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\alpha[t/y]] \end{aligned}$$

and finally we have

$$\bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\xi] \vee^{\mathbf{LT}} ([\chi] \rightarrow^{\mathbf{LT}} [\phi[t/x]]) = [\forall y(\xi \vee \alpha^\forall)] = [\xi] \vee^{\mathbf{LT}} \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} ([\chi] \rightarrow^{\mathbf{LT}} [\phi[t/x]])$$

and

$$\bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\xi] \vee^{\mathbf{LT}} ([\phi[t/x]] \rightarrow^{\mathbf{LT}} [\chi]) = [\forall y(\xi \vee \alpha^\exists)] = [\xi] \vee^{\mathbf{LT}} \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} ([\phi[t/x]] \rightarrow^{\mathbf{LT}} [\chi]).$$

□

In the propositional case, there is a *canonical* \mathcal{L}_A -evaluation over \mathbf{LT}^Γ , $\iota: \mathcal{L}_A \rightarrow \mathbf{LT}^\Gamma$, which is defined by

$$\iota(\phi) := [\phi].$$

By Lemma 5.1, ι indeed is a well-defined \mathcal{L}_A -evaluation.

In the first-order case, this Lindenbaum-Tarski algebra now forms the algebraic part of the *Lindenbaum-Tarski model*: define $\mathfrak{L}\mathfrak{T}^\Gamma$ as a model with \mathbf{LT}^Γ as the underlying Heyting algebra and with \mathcal{T}_A as the domain by setting

$$f^{\mathfrak{L}\mathfrak{T}}(t_1, \dots, t_n) := f(t_1, \dots, t_n) \text{ as well as } P^{\mathfrak{L}\mathfrak{T}}(t_1, \dots, t_n) := [P(t_1, \dots, t_n)]$$

for functions symbols f and predicate symbols P of \mathcal{L}_A . We define the *canonical variable assignment*

$$\iota: \text{Var}_A \rightarrow \mathcal{T}_A, x \mapsto x$$

and denote it, for convenience, also by ι but the context will make it clear whether we mean the propositional evaluation or the first-order variable assignment.

First note that $\mathfrak{L}\mathfrak{T}^\Gamma$ is indeed suitable for ι .

Lemma 5.2. *The model $\mathfrak{L}\mathfrak{T}^\Gamma$ is a suitable model for ι . In particular, given some $\bar{t} = (t_1, \dots, t_n) \in (\mathcal{T}_A)^n$ and $\bar{x} = (x_{i_1}, \dots, x_{i_n}) \in (\text{Var}_A)^n$ with $i_j \neq i_k$ for $j \neq k$, it holds that*

$$\overline{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}})}(\phi) = [\phi[\bar{t}/\bar{x}]].$$

for any $\phi \in \mathcal{L}_A$.

Proof. Consider $(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}})$ for \bar{x} and \bar{t} as required. By Remark 4.12, showing that $\phi \mapsto [\phi[\bar{t}/\bar{x}]]$ is the extension of $(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}})$ for those \bar{t}, \bar{x} is enough to establish suitability.

Note first that we have

$$s \left(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}} \right) = s[\bar{t}/\bar{x}] \in \mathcal{T}_A$$

for any $s \in \mathcal{T}_A$ as by assumption, \mathcal{T}_A is closed under substitution. The lemma is then proved by induction on ϕ . By the above result for terms, we get

$$[P(s_1, \dots, s_n)[\bar{t}/\bar{x}]] = [P(s_1[\bar{t}/\bar{x}], \dots, s_n[\bar{t}/\bar{x}])] = P^{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}})}(s_1^{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}})}, \dots, s_n^{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota_{\bar{x}}^{\bar{t}})}).$$

Further, we in particular have $[\perp[\bar{t}/\bar{x}]] = [\perp]$ and this provides the result for the atomic cases. The cases for $\phi \circ \psi \in \mathcal{L}_A$ with $\circ \in \{\wedge, \vee, \rightarrow\}$ follow just by noting that (simultaneous) substitution distributes over \circ .

The same also holds true for $\bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ where we have

$$\left[\left(\bigwedge_{i \in \omega} \phi_i \right) [\bar{t}/\bar{x}] \right] = \left[\bigwedge_{i \in \omega} \phi_i[\bar{t}/\bar{x}] \right] = \bigwedge_{i \in \omega}^{\mathbf{LT}} [\phi_i[\bar{t}/\bar{x}]]$$

and similarly for $\bigvee_{i \in \omega} \phi_i \in \mathcal{L}_A$ where we have used Lemma 5.1. This gives the infinitary cases.

Lastly, let $\phi \in \mathcal{L}_A$ and $x \in \text{Var}_A$. Then, we have

$$[(\forall x \phi)[\bar{t}/\bar{x}]] = [\forall z \phi[(\bar{t}', z)/(\bar{x}', x)]]$$

where z and \bar{x}', \bar{t}' are as in the definition of first-order substitutions. Using Lemma 5.1, we get

$$[\forall z \phi[(\bar{t}', z)/(\bar{x}', x)]] = \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[(\bar{t}', z)/(\bar{x}', x)][t/z]] = \bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[(\bar{t}', t)/(\bar{x}', x)]].$$

Using similar reasoning, one can show

$$[(\exists x \phi)[\bar{t}/\bar{x}]] = \bigvee_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[(\bar{t}', t)/(\bar{x}', x)]]$$

in the existential case. This gives the quantifier cases by noting that

$$\left(\iota \frac{\bar{t}}{x} \right) \frac{t}{x} = \iota \frac{(\bar{t}', t)}{(x', x)}.$$

□

This immediately yields completeness theorems for $\mathcal{G}_{\omega_1}(\mathcal{L}_A)$ or respectively $\mathcal{G}_{\omega_1, \omega}(\mathcal{L}_A)$ w.r.t. L-algebras:

Theorem 5.3. *Let \mathcal{L}_A be any fragment of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$ (with Γ closed in the first-order case), the following are equivalent:*

- (1) $\Gamma \vdash_{\mathcal{G}(\mathcal{L}_A)} \phi$;
- (2) $\Gamma \models_{\mathbf{L}(\mathcal{L}_A)} \phi$.

Here, we write \mathcal{G} for either \mathcal{G}_{ω_1} or $\mathcal{G}_{\omega_1, \omega}$, respectively.

Proof. “(1) implies (2)” is contained in Lemma 4.7 or Lemma 4.13, respectively. For the converse, suppose $\Gamma \not\vdash_{\mathcal{G}(\mathcal{L}_A)} \phi$ and construct the Lindenbaum-Tarski algebra \mathbf{LT}^Γ as indicated above.

Then Lemma 5.1 gives that ι is a well-defined \mathbf{LT}^Γ -valued \mathcal{L}_A -evaluation in the propositional case and by construction, we have $\iota[\Gamma] \subseteq \{1^{\mathbf{LT}}\}$ but $\iota(\phi) \neq 1^{\mathbf{LT}}$. Therefore $\Gamma \not\models_{\mathbf{L}(\mathcal{L}_A)} \phi$ as \mathbf{LT}^Γ is an L-algebra.

In the first-order case, we construct the Lindenbaum-Tarski model $\mathfrak{L}\mathfrak{T}^\Gamma$ as above which is suitable for ι by Lemma 5.2. Again by Lemma 5.1, the underlying algebra \mathbf{LT}^Γ is a well-defined L-algebra and for the corresponding ι , we have

$$\overline{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota)[\Gamma]} \subseteq \{1^{\mathbf{LT}}\} \text{ but } \overline{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota)}(\phi) = [\phi] \neq 1^{\mathbf{LT}}$$

by Lemma 5.2 and therefore $\Gamma \not\models_{\mathbf{L}(\mathcal{L}_A)} \phi$. □

However, restricting to countable fragments even yields a further completeness result for \mathcal{G}^D w.r.t. countable chains and thus, by Lemma 4.5, for $[0, 1]$ which is what we will outline next.

Theorem 5.4. *Let \mathcal{L}_A be a countable fragment of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$ (with Γ closed in the first-order case), the following are equivalent:*

- (1) $\Gamma \vdash_{\mathcal{G}^D(\mathcal{L}_A)} \phi$;
- (2) $\Gamma \models_{\mathbf{DL}(\mathcal{L}_A)} \phi$;
- (3) $\Gamma \models_{\mathbf{C}(\mathcal{L}_A)} \phi$;
- (4) $\Gamma \models_{\mathbf{CC}(\mathcal{L}_A)} \phi$;
- (5) $\Gamma \models_{[\mathbf{0}, \mathbf{1}]_{\mathbb{Q}}(\mathcal{L}_A)} \phi$;
- (6) $\Gamma \models_{[\mathbf{0}, \mathbf{1}]_{\mathbb{R}}(\mathcal{L}_A)} \phi$.

Here, we again write \mathcal{G}^D for either $\mathcal{G}_{\omega_1}^D$ or $\mathcal{G}_{\omega_1, \omega}^D$, respectively.

Proof. “(1) implies (2)” is contained in Lemma 4.7 or Lemma 4.13, respectively. “(2) implies (3)” follows from the fact that every chain is a distributive L-algebra. “(3) implies (4)” and “(4) implies (5)” as well as “(3) implies (6)” and “(6) implies (5)” are also immediate.

We thus only show “(5) implies (1)” and for that, suppose $\Gamma \not\vdash_{\mathcal{G}^D(\mathcal{L}_A)} \phi$. Also here, we construct the corresponding Lindenbaum-Tarski algebra \mathbf{LT}^Γ and naturally have $[\phi] \neq 1^{\mathbf{LT}}$. Lemma 5.1 again guarantees that \mathbf{LT}^Γ is a well-defined L-algebra and that ι is a well-defined evaluation in the propositional case.

In the first-order case, we again construct the Lindenbaum-Tarski model $\mathfrak{L}\mathfrak{T}^\Gamma$ over the algebra \mathbf{LT}^Γ and by Lemma 5.2, the model $\mathfrak{L}\mathfrak{T}^\Gamma$ is suitable for ι . With ι as the canonical variable assignment, we also get

$$\overline{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota)}(\psi) = [\psi] \text{ for all } \psi \in \mathcal{L}_A$$

by the same result.

Lemma 5.1 now yields that any of the meets

$$\bigwedge_{i \in \omega} ([\chi] \rightarrow^{\mathbf{LT}} [\phi_i]) \text{ and } \bigwedge_{i \in \omega} ([\psi_i] \rightarrow^{\mathbf{LT}} [\chi])$$

and additionally, in the first-order case, any of the meets

$$\bigwedge_{t \in \mathcal{T}_A} ([\chi] \rightarrow^{\mathbf{LT}} [\phi[t/x]]) \text{ and } \bigwedge_{t \in \mathcal{T}_A} ([\phi[t/x]] \rightarrow^{\mathbf{LT}} [\chi])$$

are distributive for any $\phi, \chi, \bigwedge_{i \in \omega} \phi_i, \bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$ and $x \in \text{Var}_A$.

As \mathcal{L}_A is countable, we can enumerate all of the above (distributive) meets and by Lemma 4.4, there is a prime filter F with $[\phi] \notin F$ and such that F preserves all the above meets (depending on the propositional or first-order case).

As every element of \mathbf{LT}^Γ is of the form $[\chi]$, Lemma 4.3 gives that the map

$$p_F : \mathbf{LT}^\Gamma \rightarrow \mathbf{LT}^\Gamma/F, \quad [\chi] \mapsto [\chi]_F := [[\chi]]_F$$

is a homomorphism of Heyting algebras with preserves all the meets/joins

$$\bigwedge_{i \in \omega}^{\mathbf{LT}} [\phi_i] \text{ and } \bigvee_{i \in \omega}^{\mathbf{LT}} [\psi_i]$$

as well as

$$\bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[t/x]] \text{ and } \bigvee_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[t/x]]$$

for any $\phi, \bigwedge_{i \in \omega} \phi_i, \bigvee_{i \in \omega} \psi_i \in \mathcal{L}_A$ and $x \in \text{Var}_A$. Further, by Lemma 4.1, \mathbf{LT}^Γ/F is a chain as F is a prime filter and \mathbf{LT}^Γ is an L-algebra.

In the propositional case, Lemma 4.8 implies that $p_F \circ \iota$ is a well-defined \mathcal{L}_A -evaluation into \mathbf{LT}^Γ/F with $p_F(\iota(\phi)) = [\phi]_F \neq 1^{\mathbf{LT}^\Gamma/F}$ as $[\phi] \notin F$.

In the first-order case, since $\phi[\bar{t}/\bar{x}] \in \mathcal{L}_A$ for any $\phi \in \mathcal{L}_A$, any $\bar{t} \in (\mathcal{T}_A)^n$ and any $\bar{x} \in (\text{Var}_A)^n$ with $i_j \neq i_k$ for $j \neq k$, we in particular have that the map p_F preserves the meets and joins

$$\bigwedge_{i \in \omega}^{\mathbf{LT}} [\phi_i[\bar{t}/\bar{x}]] \text{ and } \bigvee_{i \in \omega}^{\mathbf{LT}} [\psi_i[\bar{t}/\bar{x}]]$$

as well as

$$\bigwedge_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[\bar{t}/\bar{x}][t/x]] \text{ and } \bigvee_{t \in \mathcal{T}_A}^{\mathbf{LT}} [\phi[\bar{t}/\bar{x}][t/x]].$$

Since by Lemma 5.2, we have $\overline{(\mathfrak{L}\mathfrak{T}^\Gamma, \iota \frac{\bar{t}}{\bar{x}})}(\phi) = [\phi[\bar{t}/\bar{x}]]$, the map p_F fulfills the premises of Lemma 4.14 and therefore, the model $p_F \circ \mathfrak{L}\mathfrak{T}^\Gamma$ is suitable for ι with $\overline{(p_F \circ \mathfrak{L}\mathfrak{T}^\Gamma, \iota \frac{\bar{t}}{\bar{x}})}(\phi) = [\phi[\bar{t}/\bar{x}]]_F$.

Lemma 4.5 guarantees the existence of an injective Heyting algebra homomorphism

$$q : \mathbf{LT}^\Gamma/F \rightarrow [0, 1]_{\mathbb{Q}}$$

which preserves all the meets and joins existing in \mathbf{LT}^Γ/F .

In the propositional case, by Lemma 4.8 we have that $q \circ (p_F \circ \iota)$ is a well-defined \mathcal{L}_A -evaluation with $(q \circ (p_F \circ \iota))(\phi) \neq 1$ by injectivity. Naturally, we have $(q \circ (p_F \circ \iota))[\Gamma] \subseteq \{1\}$ and therefore $\Gamma \not\models_{[0,1]_{\mathbb{Q}}(\mathcal{L}_A)} \phi$.

In the first-order case, since $p_F \circ \mathfrak{L}\mathfrak{T}^\Gamma$ is suitable for ι , this implies that in particular the conditions of Lemma 4.14 are met again and therefore, the model

$$q \circ (p_F \circ \mathfrak{L}\mathfrak{T}^\Gamma)$$

is suitable for ι as well. In particular, we have

$$\overline{(q \circ (p_F \circ \mathfrak{L}\mathfrak{T}^\Gamma), \iota)}(\phi) = q([\phi]_F) \neq 1$$

as q is injective and as $[\phi] \notin F$, i.e. $[\phi]_F \neq 1^{\mathbf{LT}^\Gamma/F}$. \square

Corollary 5.5. Let \mathcal{L}_A be an arbitrary fragment of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$ but let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$ be countable with Γ closed in the first-order case. Then, the following are equivalent:

- (1) $\Gamma \vdash_{\mathcal{G}^D(\mathcal{L}_A)} \phi$;
- (2) $\Gamma \models_{\text{DL}(\mathcal{L}_A)} \phi$;
- (3) $\Gamma \models_{\text{C}(\mathcal{L}_A)} \phi$;
- (4) $\Gamma \models_{[0,1]_{\mathbb{R}}(\mathcal{L}_A)} \phi$.

Here, we again write \mathcal{G}^D for either $\mathcal{G}_{\omega_1}^D$ or $\mathcal{G}_{\omega_1, \omega}^D$, respectively.

Proof. The directions “(1) implies (2)”, “(2) implies (3)” as well as “(3) implies (4)” follow as before. Suppose $\Gamma \not\models_{\mathcal{G}^D(\mathcal{L}_A)} \phi$. Then, also

$$\Gamma \not\models_{\mathcal{G}^D(\mathcal{L}_B)} \phi$$

for $\mathcal{L}_B := \text{frag}(\Gamma \cup \{\phi\})$ since $\mathcal{L}_B \subseteq \mathcal{L}_A$. Since $\Gamma \cup \{\phi\}$ is countable, we have that \mathcal{L}_B is countable as well. Theorem 5.4 gives $\Gamma \not\models_{[0,1]_{\mathbb{R}}(\mathcal{L}_B)} \phi$.

Therefore, in the propositional case there is an \mathcal{L}_B -evaluation $v : \mathcal{L}_B \rightarrow [0, 1]$ such that $v[\Gamma] \subseteq \{1\}$ but $v(\phi) \neq 1$. Similarly, in the first-order case, there is an interpretation $\mathfrak{J} = (\mathfrak{M}, v)$ such that $\overline{\mathfrak{J}}[\Gamma] \subseteq \{1\}$ but $\overline{\mathfrak{J}}(\phi) \neq 1$.

In $[0, 1]_{\mathbb{R}}$ (by completeness), every function $v : Var_A \rightarrow [0, 1]$ has a single extension to an evaluation which we denote here by \bar{v} . If we set

$$v' : p \mapsto \begin{cases} v(p) & \text{if } p \in \mathcal{L}_B, \\ 0 & \text{otherwise,} \end{cases}$$

for $p \in Var_A$, then it is easy to see that $\bar{v}'(\psi) = v(\psi)$ for $\psi \in \mathcal{L}_B$ (as v is an \mathcal{L}_B -evaluation) and therefore, we have $\bar{v}'[\Gamma] = 1$ but $\bar{v}'(\phi) \neq 1$.

On the first-order side, any model over $[0, 1]_{\mathbb{R}}$ (again by completeness) is suitable for any variable assignment. We define \mathfrak{M}' by $f^{\mathfrak{M}'} := f^{\mathfrak{M}}$ and $P^{\mathfrak{M}'} := P^{\mathfrak{M}}$ for function symbols f and predicate symbols P in σ_B and otherwise set $f^{\mathfrak{M}'}$ and $P^{\mathfrak{M}'}$ arbitrary for symbols from $\sigma_A \setminus \sigma_B$. Further, we define $v'(x) := v(x)$ if $x \in Var_B$ and set it arbitrary otherwise for $x \in Var_A \setminus Var_B$. Again, a simple induction shows that

$$\overline{(\mathfrak{M}', v')}(\psi) = \overline{(\mathfrak{M}, v)}(\psi)$$

for any $\psi \in \mathcal{L}_B$ which results in $\Gamma \not\models_{[0,1]_{\mathbb{R}}(\mathcal{L}_A)} \phi$. \square

In particular, for countable Γ (closed, in the first-order case) in the appropriate language, we have

$$\Gamma \vdash_{\mathcal{G}_{\omega_1}^D} \phi \text{ iff } \Gamma \models_{\mathcal{G}_{\omega_1}} \phi.$$

as well as

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^D} \phi \text{ iff } \Gamma \models_{\mathcal{G}_{\omega_1, \omega}} \phi.$$

We will actually see later on that the requirement that the set of premises is countable can not be removed.

Remark 5.6. In the first-order cases, the formulations of the various completeness results require the set of premises Γ to be closed. This is, however, *only* needed for the soundness result as can be seen by inspecting the various proofs for the converse direction: both the Lindenbaum-Tarski algebra and model do not rely on Γ to be closed and neither do any remaining parts of the above presented completeness proofs.

6. HYPERSEQUENT CALCULI FOR $\mathcal{G}_{\omega_1}^D$ AND $\mathcal{G}_{\omega_1, \omega}^D$ AND CUT-ELIMINATION

We now want to address structural proof theory for infinitary Gödel logics, both for the propositional and first-order instances with operations of countable length. For that, we lift the usual approach towards structural proof theory for Gödel logics via hypersequent calculi (see [5]) to the infinitary case and provide cut-elimination theorems.

6.1. Sequents, Hypersequents and Related Notions. Hypersequents, as introduced by Avron [3], are multisets of sequents and the rules operating on hypersequents then allow for parallel modification of sequents as well as for “exchange of information” between sequents (as in particular exemplified by the rule (com) given by Avron). For the upcoming proof of the cut-elimination theorem, we introduce a version of hypersequents using *sets of formulas* for sequents and *sets of sequents* for hypersequents to avoid uses of contraction. With this set-version, we follow both Tait [33] as well as Baaz and Ciabattoni [4].

More formally, let \mathcal{L} be either \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$. A *sequent* is a pair

$$\Gamma \triangleleft \Delta$$

of finite sets Γ and Δ of \mathcal{L} -formulas where Δ contains at most one element. We use \triangleleft as a sequent delimiter to distinguish it from \Rightarrow or \vdash , which already have meanings in our context. We write Γ, Δ for the union of Γ and Δ and simply write ϕ for $\{\phi\}$. A *hypersequent* is then a set of sequents $\Gamma_i \triangleleft \Delta_i$ ($1 \leq i \leq n$) which we denote by

$$\{\Gamma_1 \triangleleft \Delta_1\} \cup \dots \cup \{\Gamma_n \triangleleft \Delta_n\}.$$

We write $\Gamma \triangleleft \Delta$ for the hypersequent $\{\Gamma \triangleleft \Delta\}$. In any other way, we follow the notational conventions of [5]. Further, there is a *canonical interpretation* \mathcal{I} (up to some fixed ordering of formulas and sequents) of hypersequents as \mathcal{L} -formulas: set $\mathcal{I}(\Gamma \triangleleft \Delta) := \bigwedge \Gamma \rightarrow \bigvee \Delta$ where $\bigwedge \Gamma$ (respectively $\bigvee \Delta$) is the conjunction (respectively disjunction) over all members of Γ (respectively Δ), with the convention that $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \perp$. This \mathcal{I} extends to hypersequents by

$$\mathcal{I}(\{\Gamma_1 \triangleleft \Delta_1\} \cup \dots \cup \{\Gamma_n \triangleleft \Delta_n\}) := \bigvee_{i=1}^n \mathcal{I}(\Gamma_i \triangleleft \Delta_i).$$

The hypersequent systems which we consider are based on a hypersequent calculus for finitary propositional Gödel logic introduced by Avron [3]. This calculus naturally extends the usual sequent calculus for intuitionistic logic, lifted to the hypersequent setting, by a specific rule emulating the prelinearity axiom scheme $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$. Avron’s calculus was extended to first-order Gödel logics by Baaz and Zach in [11].

6.2. **The Systems $\mathcal{SHG}_{\omega_1}^D$ and $\mathcal{SHG}_{\omega_1, \omega}^D$.** The range of constituting rules for the various hypersequent calculi can be seen in Figure 1. They consist of the rules given in [4] (modified for the set-based hypersequents) for a hypersequent calculus for first-order Gödel logic extended by four additional infinitary rules. Following Baaz and Ciabattoni [4], the version of (com) given here differs from the one usually given (see e.g. [11]) and, similar to [4], this serves some technical purposes in the following cut-elimination proof. For further context, see in particular Remark 2 from [4]. Principal formulas are defined as commonly done. The rules (\forall, l) and

$$\begin{array}{c}
\text{Initial Hypersequents} \\
\frac{}{\phi \triangleleft \phi}, \phi \neq \perp \text{ and atomic (id)} \quad \frac{}{\perp \triangleleft} (\perp) \\
\text{Structural Rules} \\
\frac{G}{G \cup \{\Gamma \triangleleft \Delta\}} \text{ (EW)} \quad \frac{G \cup \{\Gamma_1, \Gamma_2 \triangleleft \Delta_1\} \quad G \cup \{\Gamma_1, \Gamma_2 \triangleleft \Delta_2\}}{G \cup \{\Gamma_1 \triangleleft \Delta_1\} \cup \{\Gamma_2 \triangleleft \Delta_2\}} \text{ (com)} \\
\frac{G \cup \{\Gamma \triangleleft \Delta\}}{G \cup \{\Gamma, \phi \triangleleft \Delta\}} \text{ (w, l)} \quad \frac{G \cup \{\Gamma \triangleleft\}}{G \cup \{\Gamma \triangleleft \phi\}} \text{ (w, r)} \\
\frac{G \cup \{\Gamma, \phi \triangleleft \Delta\} \quad G \cup \{\Gamma \triangleleft \phi\}}{G \cup \{\Gamma \triangleleft \Delta\}} \text{ (cut)} \\
\text{Logical Rules} \\
\frac{G \cup \{\Gamma \triangleleft \phi\} \quad G \cup \{\Gamma, \psi \triangleleft \Delta\}}{G \cup \{\Gamma, \phi \rightarrow \psi \triangleleft \Delta\}} (\rightarrow, l) \quad \frac{G \cup \{\Gamma, \phi \triangleleft \psi\}}{G \cup \{\Gamma \triangleleft \phi \rightarrow \psi\}} (\rightarrow, r) \\
\frac{G \cup \{\Gamma, \phi_i \triangleleft \Delta\}}{G \cup \{\Gamma, \phi_0 \wedge \phi_1 \triangleleft \Delta\}} (\wedge, l)_{i=0,1} \quad \frac{G \cup \{\Gamma \triangleleft \phi\} \quad G \cup \{\Gamma \triangleleft \psi\}}{G \cup \{\Gamma \triangleleft \phi \wedge \psi\}} (\wedge, r) \\
\frac{G \cup \{\Gamma, \phi \triangleleft \Delta\} \quad G \cup \{\Gamma, \psi \triangleleft \Delta\}}{G \cup \{\Gamma, \phi \vee \psi \triangleleft \Delta\}} (\vee, l) \quad \frac{G \cup \{\Gamma \triangleleft \phi_i\}}{G \cup \{\Gamma \triangleleft \phi_0 \vee \phi_1\}} (\vee, r)_{i=0,1} \\
\text{Quantifier Rules} \\
\frac{G \cup \{\phi[t/x], \Gamma \triangleleft \Delta\}}{G \cup \{\forall x \phi(x), \Gamma \triangleleft \Delta\}} (\forall, l) \quad \frac{G \cup \{\Gamma \triangleleft \phi[a/x]\}}{G \cup \{\Gamma \triangleleft \forall x \phi(x)\}} (\forall, r) \\
\frac{G \cup \{\phi[a/x], \Gamma \triangleleft \Delta\}}{G \cup \{\exists x \phi(x), \Gamma \triangleleft \Delta\}} (\exists, l) \quad \frac{G \cup \{\Gamma \triangleleft \phi[t/x]\}}{G \cup \{\Gamma \triangleleft \exists x \phi(x)\}} (\exists, r) \\
\text{Infinitary Rules} \\
\frac{G \cup \{\phi_j, \Gamma \triangleleft \Delta\}}{G \cup \{\bigwedge_{i \in \omega} \phi_i, \Gamma \triangleleft \Delta\}} (\bigwedge, l) \quad \frac{G \cup \{\Gamma \triangleleft \phi_i\} (i \in \omega)}{G \cup \{\Gamma \triangleleft \bigwedge_{i \in \omega} \phi_i\}} (\bigwedge, r) \\
\frac{G \cup \{\phi_i, \Gamma \triangleleft \Delta\} (i \in \omega)}{G \cup \{\bigvee_{i \in \omega} \phi_i, \Gamma \triangleleft \Delta\}} (\bigvee, l) \quad \frac{G \cup \{\Gamma \triangleleft \phi_j\}}{G \cup \{\Gamma \triangleleft \bigvee_{i \in \omega} \phi_i\}} (\bigvee, r)
\end{array}$$

FIGURE 1. The various hypersequent rules.

(\exists, r) are supposed to fulfill the eigenvariable condition: the variable a is free and does not occur in the lower hypersequent. We refer with $\mathcal{SHG}_{\omega_1}^D$ to all initial, structural, logical and infinitary rules (over the propositional language) and with and $\mathcal{SHG}_{\omega_1, \omega}^D$ to $\mathcal{SHG}_{\omega_1}^D$ (now over the first-order language) extended with the quantifier rules.

Deductions in the hypersequent calculi are defined by natural infinitary generalizations of the usual definition: deductions are countable (possibly infinite) well-founded rooted trees where every node is labeled with a hypersequent and every edge is labeled with a rule such that the arities of the rules are respected and the applications are correct. If d is such derivation with a root hypersequent H , then we write $d \vdash^s_{\mathcal{SHG}_{\omega_1, \omega}^D} H$ or $d \vdash^s_{\mathcal{SHG}_{\omega_1}^D} H$ (depending on the used language and systems). We omit the system if it is arbitrary or clear from the context and then write $d \vdash^s H$. We write $\vdash^s H$ if there is any derivation d with $d \vdash^s H$. Also, derivations with additional hypersequent assumptions are defined as always. The notion of substitution naturally carries over to sequents and hypersequents as well as derivations and we use the same notation as in formulas and terms.

One quickly verifies that the non-atomic version of (id), i.e. allowing for *arbitrary* ϕ , is derivable in the systems. Further, they are complete w.r.t. to the Hilbert-type systems introduced before in the following sense:

Theorem 6.1. For any $\phi \in \mathcal{L}_{\omega_1}$, $\vdash_{\mathcal{SHG}_{\omega_1}^D} \triangleleft \phi$ if, and only if $\vdash_{\mathcal{G}_{\omega_1}^D} \phi$. Similarly for $\mathcal{SHG}_{\omega_1, \omega}^D$ and $\mathcal{G}_{\omega_1, \omega}^D$ over $\mathcal{L}_{\omega_1, \omega}$.

Proof. The only thing we want to remark is that the infinitary distributivity axiom (D) is derivable in the calculus, akin to the analogous derivation of the first order axiom (QS) e.g. given in [5] (where we omit weakenings in the derivation):

$$\frac{\frac{\frac{\phi \triangleleft \phi \quad \psi_i \triangleleft \psi_i}{\{\phi \triangleleft \psi_i\} \cup \{\psi_i \triangleleft \phi\}} \text{(com)}}{\{\phi \triangleleft \psi_i\} \cup \{\phi \vee \psi_i \triangleleft \phi\}} \text{(}\vee, \text{l)}}{\frac{\frac{\phi \triangleleft \phi}{\{\phi \vee \psi_i \triangleleft \psi_i\} \cup \{\phi \vee \psi_i \triangleleft \phi\}} \text{(}\vee, \text{l)}}{\{\bigwedge_{i \in \omega} (\phi \vee \psi_i) \triangleleft \psi_i\} \cup \{\bigwedge_{i \in \omega} (\phi \vee \psi_i) \triangleleft \phi\}} 2 \times \text{(}\wedge, \text{l)}} \text{(}\vee, \text{l)}$$

for any $i \in \omega$. Using those as input for (\wedge, r) , we get

$$\frac{\frac{\frac{\vdots}{\{\bigwedge_{i \in \omega} (\phi \vee \psi_i) \triangleleft \bigwedge_{i \in \omega} \psi_i\} \cup \{\bigwedge_{i \in \omega} (\phi \vee \psi_i) \triangleleft \phi\}} \text{(}\wedge, \text{r)}}{\bigwedge_{i \in \omega} (\phi \vee \psi_i) \triangleleft \phi \vee \bigwedge_{i \in \omega} \psi_i} \text{(}\vee_0, \text{r), (}\vee_1, \text{r)}}{\triangleleft \bigwedge_{i \in \omega} (\phi \vee \psi_i) \rightarrow \phi \vee \bigwedge_{i \in \omega} \psi_i} \text{(}\rightarrow, \text{r)}.$$

□

6.3. Cut-Elimination in the Schütte-Tait Style. We now turn to cut-elimination. Most cut-elimination methods fall into one of two categories: Gentzen style methods [16] which remove highest cuts and Schütte-Tait style methods [30, 33] which remove most complex cuts (in the sense of occurring logical symbols or a similar complexity measure). Using highest cuts does not generally result in terminating procedures with systems which have infinitary rules and we thus opt for a Schütte-Tait style proof.

For proving cut-elimination, we closely follow the argument given in [4] by Baaz and Ciabattoni where the authors provide a Schütte-Tait style cut-elimination proof for the calculus which we have used (together with its propositional fragment) as a basis for our infinitary extensions. Naturally, the finitary notions used there have to be appropriately extended to arbitrary countable ordinals and we do this in a similar vein as in Tait's work [33].

Before moving on to the technical results, we will need to introduce various measures on proofs and formulas. For this, we first give a short primer on the relevant notions regarding ordinals. For a general overview over ordinal arithmetic and further notions, see [32]. We write $+$ and \cdot for the usual ordinal addition and multiplication. Given a family of ordinals α_i , we write $\sup_i^+ \alpha_i$ for the smallest ordinal greater than every α_i and $\sup_i \alpha_i$ for the smallest ordinal greater or equal than every α_i . A function on ordinals is called *continuous* if it commutes with \sup (although we will not really rely on this notion).

Further, we will need the *natural sum* and *multiplication* on ordinals, also called the Hessenberg sum and multiplication (see [32]), which we denote by $x \oplus y$ and $x \otimes y$, respectively. The precise definition for \oplus and \otimes can be given using Cantor normal forms, among others, but we will only need certain properties of \oplus , \otimes and their interplay with \sup^+ which we list in the following remark.

Remark 6.2. \oplus and \otimes are commutative, associative and monotone in both arguments. Further, we have $\alpha \oplus 1 = \alpha + 1$ for any ordinal α and $\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma)$. We have $\sup_i^{(+)}(\beta \oplus \alpha_i) \leq \beta \oplus \sup_i^{(+)} \alpha_i$ and $\sup_i^{(+)}(\beta \otimes \alpha_i) \leq \beta \otimes \sup_i^{(+)} \alpha_i$ where $\sup^{(+)}$ is either \sup or \sup^+ .

There is a notion of exponentiation derived from natural multiplication, first considered by de Jongh and Parikh [14], which we denote by $\alpha^{\otimes \beta}$ and call (*super-Jacobsthal*) *exponentiation*, following Altman [2]. This exponentiation can be formally defined using transfinite recursion by

- (1) $\alpha^{\otimes 0} := 1$ for any α ,
- (2) $\alpha^{\otimes(\beta+1)} := \alpha^{\otimes \beta} \otimes \alpha$ for any α, β ,
- (3) $\alpha^{\otimes \beta} := \sup_{\gamma < \beta} \alpha^{\otimes \gamma}$ for any α and any limit ordinal β ,

and has, in particular, the following properties: $\alpha^{\otimes \beta}$ is strictly increasing and continuous in β and

$$\alpha^{\otimes(\beta \oplus \gamma)} = \alpha^{\otimes \beta} \otimes \alpha^{\otimes \gamma}.$$

The first two are immediate by the monotonicity of \otimes and the definition of exponentiation. A proof of the latter can be found in [2].

Inspired by Tait [33], we define the function $\chi^0(\alpha) := 4^{\otimes \alpha}$ and derived from that, we define χ^z as the function enumerating the common fixed points of χ^w for all $w < z$. These χ^z are the Veblen iterations of χ^0 which exist (for all countable ordinals) by [38] as χ^0 is continuous and strictly increasing. Further, all χ^z are also continuous and strictly increasing and this latter property is all that will be needed in the following regarding χ^z , besides its definition using fixed points.

We define the *complexity* $|\phi|$ of $\phi \in \mathcal{L}_{\omega_1}$ recursively by

- (1) $|\phi| := 0$ for atomic ϕ ,
- (2) $|\phi \circ \psi| := \max\{|\phi|, |\psi|\} + 1$,
- (3) $|\bigwedge_{i \in \omega} \phi_i| := |\bigvee_{i \in \omega} \phi_i| := \sup_{i \in \omega}^+ |\phi_i|$.

$|\cdot|$ is extended to $\mathcal{L}_{\omega_1, \omega}$ by adding the clause

- (4) $|Qx\phi| := |\phi| + 1$,

for $Q \in \{\forall, \exists\}$.

Let d be a derivation with d_i , $i < k \leq \omega$, as its direct predecessors (i.e., those subderivations proving the assumptions of the last rule in d , also called *direct subderivations* in [33]). As natural generalizations of the notions defined in [4] (and in some way akin to [33]) we define the *rank* (or *height*) $|d|$ of the derivation (without counting weakenings), $|d|$, by setting $|d| := \sup_{i < k}^+ |d_i|$ if the last rule was not a weakening and $|d| := |d_0|$ otherwise. To complement this, we define the height of the internal weakenings $w(d)$ by $w(d) := \sup_{i < k} w(d_i)$ if the last rule was not an internal weakening and $w(d) := w(d_0) + 1$ otherwise. Similarly, we define $W(d)$ by using external instead of internal weakenings.

Lastly, we recursively define the *cut-degree* of d , $\rho(d)$, by

- (1) $\rho(d) := 0$ if d is cut-free,
- (2) $\rho(d) := \sup_{i < k} \rho(d_i)$ if the last inference is not a cut,
- (3) $\rho(d) := \max\{|\phi| + 1, \rho(d_0), \rho(d_1)\}$ if the last inference is a cut with cut-formula ϕ .

Intuitively, $\rho(d)$ is an upper bound on the complexity of all cut-formulas in d .

Lemma 6.3. *If $d \vdash^s H$, then $d[t/x] \vdash^s H[t/x]$ with $|d[t/x]| = |d|$, $w(d[t/x]) = w(d)$, $W(d[t/x]) = W(d)$ and $\rho(d[t/x]) = \rho(d)$.*

It is straightforward to check that $d[t/x]$ is a correct proof since d is. Both $d[t/x] \vdash^s H[t/x]$ and the other properties are then immediate.

The proofs which we give in the following rely on a (formal) tracking of the cut formula through the proof based on so-called *decorations* as introduced by Baaz and Ciabattoni in [4], extended to the infinitary case.

Definition 6.4. Given $d \vdash^s H$ and given a *decoration* of H , that is H where some (but not necessarily all) occurrences of a formula ϕ are decorated, denoted by ϕ^* , the decorated version of d is defined by recursion on the tree: if we have a decoration of an occurring hypersequent H' , then the premises are decorated according to which rule was used to derive H' . The definitions for the rules (EW), (w, l), (w, r), (com) and (com) are exactly as in [4]. Suppose the rule used is a (possibly infinitary) logical rule with arity $k \leq \omega$, i.e. we have

$$\frac{G \cup \{\Gamma_i \triangleleft \Delta_i \mid i < k\}}{G \cup \{\Gamma \triangleleft \Delta\}}.$$

Then if

- (1) ϕ is the principal formula of the rule: if $\phi^* \in \Gamma$ then ϕ is decorated in Γ_i iff ϕ already occurs in Γ_i .
- (2) ϕ is not the principal formula of the rule then ϕ is decorated in Γ_i or Δ_i iff it is decorated in Γ or Δ , respectively.

Further, in both cases, $G \setminus \{\Gamma_i \triangleleft \Delta_i \mid i < k\}$ is decorated as in the conclusion.

Lemma 6.5. *The following inversions are valid:*

- (i) If $d \vdash^s G \cup \{\Gamma, \phi \vee \psi \triangleleft \Delta\}$, then there are proofs $d_0 \vdash^s G \cup \{\Gamma, \phi \triangleleft \Delta\}$ and $d_1 \vdash^s G \cup \{\Gamma, \psi \triangleleft \Delta\}$.
- (ii) If $d \vdash^s G \cup \{\Gamma, \phi \wedge \psi \triangleleft \Delta\}$, then there is a proof $d_0 \vdash^s G \cup \{\Gamma, \phi, \psi \triangleleft \Delta\}$.
- (iii) If $d \vdash^s G \cup \{\Gamma \triangleleft \phi \wedge \psi\}$, then there are proofs $d_0 \vdash^s G \cup \{\Gamma \triangleleft \phi\}$ and $d_1 \vdash^s G \cup \{\Gamma \triangleleft \psi\}$.
- (iv) If $d \vdash^s G \cup \{\Gamma \triangleleft \phi \rightarrow \psi\}$, then there is a proof $d_0 \vdash^s G \cup \{\Gamma, \phi \triangleleft \psi\}$.
- (v) If $d \vdash^s G \cup \{\Gamma, \exists x \phi \triangleleft \Delta\}$, then there is a proof $d_0 \vdash^s G \cup \{\Gamma, \phi[a/x] \triangleleft \Delta\}$.
- (vi) If $d \vdash^s G \cup \{\Gamma \triangleleft \forall x \phi\}$, then there is a proof $d_0 \vdash^s G \cup \{\Gamma \triangleleft \phi[a/x]\}$.
- (vii) If $d \vdash^s G \cup \{\Gamma, \bigvee_{i \in \omega} \phi_i \triangleleft \Delta\}$, then there are proofs $d_j \vdash^s G \cup \{\Gamma, \phi_j \triangleleft \Delta\}$ for each $j \in \omega$.
- (viii) If $d \vdash^s G \cup \{\Gamma \triangleleft \bigwedge_{i \in \omega} \phi_i\}$, then there are proofs $d_j \vdash^s G \cup \{\Gamma \triangleleft \phi_j\}$ for each $j \in \omega$.

In any case, we respectively have $\rho(d_i) \leq \rho(d)$ and $|d_i| \leq |d|$.

Proof. The proof for items (i) to (vi) follows exactly the reasoning of [4] for the respective finitary result (see Lemma 4 there). We give the proofs for (vii) and (viii) in the same spirit.

- (vii) We consider a decoration of d starting with $G \cup \{\Gamma, (\bigvee_{i \in \omega} \phi_i)^* \triangleleft \Delta\}$. Replace every occurring $\{\Gamma', (\bigvee_{i \in \omega} \phi_i)^* \triangleleft \Delta'\}$ by $\{\Gamma', \phi_j \triangleleft \Delta'\}$. Delete all the subderivations but the j -th one above any application of (\bigvee, l) where $(\bigvee_{i \in \omega} \phi_i)^*$ occurs as a decorated formula and is principle. As all initial hypersequents are atomic, correctness of the resulting d_j can be shown by an induction over $|d| \oplus w(d) \oplus W(d)$. Clearly $|d_j| \leq |d|$ and $\rho(d_j) \leq \rho(d)$.
- (viii) We consider a decoration of d starting from $G \cup \{\Gamma \triangleleft (\bigwedge_{i \in \omega} \phi_i)^*\}$. Replace every occurring $\{\Gamma' \triangleleft (\bigwedge_{i \in \omega} \phi_i)^*\}$ by $\{\Gamma' \triangleleft \phi_j\}$ and delete all subderivations but the j -th one above any application of (\bigwedge, r) in which $(\bigwedge_{i \in \omega} \phi_i)^*$ occurs decorated and is principle. Again, the correctness of d_j follows by a straightforward induction on $|d| \oplus w(d) \oplus W(d)$ and we have $|d_j| \leq |d|$ and $\rho(d_j) \leq \rho(d)$ already by construction. \square

Lemma 6.6. *Suppose $d \vdash^s G \cup \{\Gamma, \phi \triangleleft \Delta\}$ where ϕ is atomic and not the cut-formula of any cut in d . Then for any Σ , there is a d' with assumption $G \cup \{\Sigma \triangleleft \phi\}$ such that $d' \vdash^s G \cup \{\Gamma, \Sigma \triangleleft \Delta\}$ with $|d'| \leq |d|$ and $\rho(d') \leq \rho(d)$.*

Proof. The proof can be easily obtained by generalizing the proof of the respective finitary result from [4]: Decorate d , starting from $G \cup \{\Gamma, \phi^* \triangleleft \Delta\}$, replace any occurring $\{\Gamma', \phi^* \triangleleft \Delta'\}$ by $\{\Gamma', \Sigma \triangleleft \Delta'\}$ and add G to every hypersequent. This tree now needs to be corrected to yield a correct proof. As in [4], weakenings which produce decorated ϕ^* are replaced by (potentially more) weakenings producing Σ . Initial sequents

$$\frac{}{\psi \triangleleft \psi} (\text{id}) \text{ and } \frac{}{\perp \triangleleft} (\perp)$$

which don't introduce ϕ^* are respectively replaced by

$$\frac{\frac{}{\psi \triangleleft \psi}}{G \cup \{\psi \triangleleft \psi\}} \text{ and } \frac{\frac{}{\perp \triangleleft}}{G \cup \{\perp \triangleleft\}}$$

where we added sufficiently many weakenings to introduce G , while initial sequents introducing ϕ^* are replaced by $G \cup \{\Sigma \triangleleft \phi\}$ which is an allowed assumption for d' . That proof is now correct as ϕ is not the cut-formula of any cut in d . As weakenings do not lengthen $|d|$, we get $|d'| \leq |d|$ and as we didn't introduce any new cut, we get $\rho(d') \leq \rho(d)$. \square

Lemma 6.7. *Suppose $d_0 \vdash^s G \cup \{\Gamma \triangleleft \phi\}$ and $d_1 \vdash^s G \cup \{\Gamma, \phi \triangleleft \Delta\}$ with $\rho(d_i) \leq |\phi|$ for $i = 0, 1$. Then there is a $d \vdash^s G \cup \{\Gamma \triangleleft \Delta\}$ with $\rho(d) \leq |\phi|$ and $|d| \leq 2 \otimes (|d_0| \oplus |d_1|)$.*

Proof. The proof is a natural extension of the corresponding finitary result from [4]. Note that since $\rho(d_i) \leq |\phi|$, ϕ is not the cut-formula of any cut in d_i .

For $\phi = \perp$, decorate d_0 starting from $G \cup \{\Gamma \triangleleft \perp^*\}$ and replace any $\{\Gamma' \triangleleft \perp^*\}$ by $\{\Gamma' \triangleleft \Delta\}$. We now have to correct the proof at the points where \perp^* originates. In this simple case, \perp^* arises by either an internal or external weakening. The internal weakenings can be either removed if Δ is empty or replaced by an internal weakening with Δ if Δ is nonempty. Similarly, the external weakenings get appropriately replaced.

For atomic $\phi \neq \perp$, note first that $\rho(d_i) \leq |\phi| = 0$ implies that d_1 and d_0 are cut-free. In particular, ϕ is not the cut-formula of any cut in d_1 and thus, there is a derivation d'_1 with assumption $G \cup \{\Gamma \triangleleft \phi\}$ such that

$$d'_1 \vdash^s G \cup \{\Gamma \triangleleft \Delta\}$$

as well as $|d'_1| \leq |d_1|$ and $\rho(d'_1) = 0$. We form d by replacing every assumption $G \cup \{\Gamma \triangleleft \phi\}$ with the proof d_0 . It is straightforward to check that $|d| \leq |d_0| \oplus |d_1| \leq 2 \otimes (|d_0| \oplus |d_1|)$ and we have $\rho(d) = 0$ by Lemma 6.6.

For $\phi = \bigvee_{i \in \omega} \phi_i$, consider a decoration of d_0 starting from $G \cup \{\Gamma \triangleleft (\bigvee_{i \in \omega} \phi_i)^*\}$. Replace any occurring $\{\Sigma \triangleleft (\bigvee_{i \in \omega} \phi_i)^*\}$ by $\{\Gamma, \Sigma \triangleleft \Delta\}$ and add G to every hypersequent and Γ to every premise of any sequent. This resulting tree is not a correct proof anymore and we consider the following correction steps on the initial rules and on the rules which introduce a decorated instance of ϕ :

- (1) Replace any initial rule

$$\frac{}{\psi \triangleleft \psi} (\text{id}) \text{ or } \frac{}{\perp \triangleleft} (\perp)$$

by

$$\frac{\frac{}{\psi \triangleleft \psi} (\text{id})}{G \cup \{\Gamma, \psi \triangleleft \psi\}} \text{ or } \frac{\frac{}{\perp \triangleleft} (\perp)}{G \cup \{\Gamma, \perp \triangleleft\}},$$

respectively, using sufficiently many applications of (EW).

- (2) Suppose $(\bigvee_{i \in \omega} \phi_i)^*$ originates as the principal formula of a logical rule. Then, replace any part of the form

$$\frac{\frac{\vdots}{G' \cup \{\Gamma' \triangleleft \phi_j\}}}{G' \cup \{\Gamma' \triangleleft (\bigvee_{i \in \omega} \phi_i)^*\}} (\vee, r)$$

occurring in the decorated version of d_0 by

$$\frac{\frac{\vdots}{G \cup (\overline{G'} \cup \{\Gamma, \Gamma' \triangleleft \phi_j\})[\Delta / (\bigvee_{i \in \omega} \phi_i)^*]} \quad \frac{\vdots d_1^{-1}}{G \cup \overline{G'} \cup \{\Gamma, \Gamma', \phi_j \triangleleft \Delta\}}}{G \cup \overline{G'} \cup \{\Gamma, \Gamma' \triangleleft \Delta\}} (\text{cut})$$

where d_1^{-1} is obtained by Lemma 6.5 from $d_1 \vdash G \cup \{\Gamma, \bigvee_{i \in \omega} \phi_i \triangleleft \Delta\}$. Here, $\overline{G'}$ is the hypersequent G' after the possible internal replacements. In the above presentation, we have suppressed the needed external and internal weakenings as they have no effect on the resulting rank.

- (3) If $(\bigvee_{i \in \omega} \phi_i)^*$ originates from an internal weakening, replace any such

$$\frac{\frac{\vdots}{G' \cup \{\Gamma' \triangleleft\}}}{G' \cup \{\Gamma' \triangleleft (\bigvee_{i \in \omega} \phi_i)^*\}} (\text{w}, r)$$

by

$$\frac{\frac{\vdots}{\overline{G'} \cup G \cup \{\Gamma' \triangleleft\}}}{\overline{G'} \cup G \cup \{\Gamma, \Gamma' \triangleleft \Delta\}} (\text{w}, r)$$

if Δ is nonempty and remove them otherwise.

- (4) If $(\bigvee_{i \in \omega} \phi_i)^*$ originates from an external weakening, replace any

$$\frac{\frac{\vdots}{G'}}{G' \cup \{\Gamma' \triangleleft (\bigvee_{i \in \omega} \phi_i)^*\}} (\text{EW})$$

by

$$\frac{\frac{\vdots}{\overline{G'} \cup G}}{\overline{G'} \cup G \cup \{\Gamma, \Gamma' \triangleleft \Delta\}} (\text{EW}).$$

The resulting proof d is correct as can be verified by transfinite induction on $|T| \oplus w(T) \oplus W(T)$ for subderivations T of d_0 . Here, it is important that ϕ is not the cut-formula of any cut.

Furthermore, any newly introduced cut uses one of the ϕ_i as a cut-formula. We thus get, using $\rho(d_1^{-1}) \leq \rho(d_1)$ from Lemma 6.5, that

$$\rho(d) \leq \max \left\{ \rho(d_0), \rho(d_1), \sup_{i \in \omega} (|\phi_i| + 1) \right\} = \max \left\{ \rho(d_0), \rho(d_1), \sup_{i \in \omega}^+ |\phi_i| \right\} \leq |\phi|$$

as $\rho(d_i) \leq |\phi|$ for $i = 0, 1$ by assumption and $|\phi| = \sup_{i \in \omega}^+ |\phi_i|$ by definition.

Regarding the rank, let T' be the replacement of any rooted sub derivation T of d_0 after the replacement and correction procedure. Then we can prove, by transfinite induction on $|T| \oplus w(T) \oplus W(T)$, that $|T'| \leq 2 \otimes (|d_1^{-1}| \oplus |T|)$.

- (1) If T is just a single initial rule, then $|T'| = |T|$ as only weakenings were added.
- (2) If T ends with an application of (\vee, r) which introduces ϕ^* in the annotation, let T_0 be its preceding derivation. Note that T' ends with an application of cut with preceding derivations T'_0 and d_1^{-1} (modulo

additional weakenings). We get

$$\begin{aligned} |T'| &= \max\{|T'_0|, |d_1^{-1}|\} + 1 \\ &\leq \max\{2 \otimes (|T_0| \oplus |d_1^{-1}|), |d_1^{-1}|\} + 1 \\ &\leq 2 \otimes (|T_0| + 1) \oplus |d_1^{-1}| \\ &= 2 \otimes (|T| \oplus |d_1^{-1}|) \end{aligned}$$

which completes this case. Here, the first inequality follows from the induction hypothesis.

- (3) If T ends with a weakening which introduces ϕ^* , then we still have $|T_0| \oplus w(T_0) \oplus W(T_0) < |T| \oplus w(T) \oplus W(T)$ and thus, we get

$$|T'| = |T'_0| \leq 2 \otimes (|d_1^{-1}| \oplus |T_0|) = 2 \otimes (|d_1^{-1}| \oplus |T|)$$

as by definition $|T_0| = |T|$ and as only weakenings were added.

- (4) If the last rule of T does not introduce ϕ^* and is not a weakening, then for the preceding derivations T'_i of T' , we have

$$|T'| = \sup_{i \in \omega}^+ |T'_i| \leq \sup_{i \in \omega}^+ 2 \otimes (|d_1^{-1}| \oplus |T_i|) \leq 2 \otimes (|d_1^{-1}| \oplus \sup_{i \in \omega}^+ |T_i|) = 2 \otimes (|d_1^{-1}| \oplus |T|).$$

- (5) If the last rule does not introduce ϕ^* but is a weakening, then the reasoning is as in case (3).

We in particular thus have (taking $d = d'_0$)

$$|d| \leq 2 \otimes (|d_1^{-1}| \oplus |d_0|) \leq 2 \otimes (|d_1| \oplus |d_0|)$$

as $|d_1| \leq |d_1^{-1}|$ by Lemma 6.5.

For $\phi = \bigwedge_{i \in \omega} \phi_i$, consider a decoration of d_1 starting from $G \cup \{\Gamma, (\bigwedge_{i \in \omega} \phi_i)^* \triangleleft \Delta\}$. Replace any occurring $\{\Gamma', (\bigwedge_{i \in \omega} \phi_i)^* \triangleleft \Delta'\}$ by $\{\Gamma', \Gamma \triangleleft \Delta'\}$, add G to every hypersequent and add Γ to the premise of every sequent. This resulting tree is not a correct proof anymore and we consider the following correction steps:

- (1) For initial rules, this is the same as the previous correction step (1).
- (2) If $(\bigwedge_{i \in \omega} \phi_i)^*$ originates as the principal formula of a logical rule, then replace any part of the form

$$\frac{\vdots}{\frac{G' \cup \{\Gamma', \phi_j \triangleleft \Delta'\}}{G' \cup \{\Gamma', (\bigwedge_{i \in \omega} \phi_i)^* \triangleleft \Delta'\}} (\vee, r)}$$

of the current proof by

$$\frac{\frac{\vdots}{G \cup (\overline{G'} \cup \{\Gamma, \Gamma', \phi_j \triangleleft \Delta'\})[\Gamma / (\bigwedge_{i \in \omega} \phi_i)^*]}{\quad} \quad \frac{\vdots d_0^{-1}}{G \cup \overline{G'} \cup \{\Gamma, \Gamma' \triangleleft \phi_j\}}}{G \cup \overline{G'} \cup \{\Gamma, \Gamma' \triangleleft \Delta\}} (\text{cut})$$

where d_0^{-1} is obtained via Lemma 6.5 from $d_0 \vdash G \cup \{\Gamma \triangleleft \bigwedge_{i \in \omega} \phi_i\}$.

- (3) If $(\bigwedge_{i \in \omega} \phi_i)^*$ originates from an internal weakening, replace any such

$$\frac{\vdots}{\frac{G' \cup \{\Gamma' \triangleleft \Delta'\}}{G' \cup \{\Gamma', (\bigwedge_{i \in \omega} \phi_i)^* \triangleleft \Delta'\}} (\text{w}, r)}$$

by

$$\frac{\vdots}{\frac{\overline{G'} \cup G \cup \{\Gamma' \triangleleft \Delta'\}}{\overline{G'} \cup G \cup \{\Gamma', \Gamma \triangleleft \Delta'\}}}$$

using stepwise internal weakenings for members of Γ .

- (4) If $(\bigwedge_{i \in \omega} \phi_i)^*$ originates from an external weakening, replace any

$$\frac{\vdots}{\frac{G'}{G' \cup \{\Gamma', (\bigwedge_{i \in \omega} \phi_i)^* \triangleleft \Delta'\}} (\text{EW})}$$

by

$$\frac{\vdots}{\overline{G' \cup G}} \text{ (EW)} \\ \overline{G' \cup G \cup \{\Gamma', \Gamma \triangleleft \Delta'\}}$$

As ϕ is again not the cut-formula of any cut in d_1 , one can verify the correctness of the resulting proof d by transfinite induction on $|T| \oplus w(T) \oplus W(T)$ of subderivations T of d_1 .

Using the same reasoning as before, we also again derive $\rho(d) \leq |\phi|$ as well as $|T'| \leq 2 \otimes (|d_0^{-1}| \oplus |T|)$ where T' is the replacement of any rooted subderivation T of d_1 after the replacement and correction procedure, as before. The latter implies $|d| \leq 2 \otimes (|d_0| \oplus |d_1|)$ as before.

The cases of $\phi = \phi_0 \vee \phi_1$ and $\phi = \exists x \phi_0$ are similar to that of $\phi = \bigvee_{i \in \omega} \phi_i$ (see [4] for the latter). The quantifier case in particular uses Lemma 6.3.

Similarly, the cases of $\phi = \phi_0 \wedge \phi_1$, $\phi = \forall x \phi_0$ and $\phi = \phi_0 \rightarrow \phi_1$ are analogous to that of $\phi = \bigwedge_{i \in \omega} \phi_i$ (see again [4] for the last). As with \exists , the \forall -case uses Lemma 6.3. \square

Lemma 6.8. *Let $d \vdash^s H$ with $\rho(d) \leq v + \omega^z$. Then there is a derivation $d' \vdash^s H$ with $|d'| \leq \chi^z(|d|)$ and $\rho(d') \leq v$.*

Proof. The theorem is proved by induction on the lexicographically ordered pair $(z, |d| \oplus w(d) \oplus W(d))$. So assume the claim for any $(\hat{z}, |\hat{d}| \oplus w(\hat{d}) \oplus W(\hat{d}))$ with $\hat{z} < z$ or $\hat{z} = z$ and $|\hat{d}| \oplus w(\hat{d}) \oplus W(\hat{d}) < |d| \oplus w(d) \oplus W(d)$. We divide the proof on whether the last inference rule of d was a cut, a weakening or neither.

Suppose the last rule was not a cut and not a weakening. Let $k \leq \omega$ be the arity of the last rule and let d_i , $i < k$, be the direct predecessors with $d_i \vdash^s H_i$. Naturally, $|d_i| < |d|$ for all $i < k$ as the last rule was not a weakening and also

$$v + \omega^z \geq \rho(d) \geq \rho(d_i)$$

for all $i < k$ by definition. Using the induction hypothesis on d_i , we get derivations $d'_i \vdash^s H_i$ with $|d'_i| \leq \chi^z(|d_i|)$ and $\rho(d'_i) \leq v$. Using the same last rule as in d , we combine the d'_i to a proof $d' \vdash H$. First, we have $\rho(d') \leq v$ as the last rule was not a cut. Further, we get

$$|d'_i| \leq \chi^z(|d_i|) < \chi^z(|d|)$$

using that $|d_i| < |d|$ and that χ^z is increasing. As $|d'|$ is the least ordinal α with $|d'_i| < \alpha$ for all i , as the last rule was not a weakening, we get $|d'| \leq \chi^z(|d|)$.

Suppose the last rule was a cut. Then we get two preceding derivations

$$d_0 \vdash^s G \cup \{\Gamma \triangleleft \phi\} \text{ and } d_1 \vdash^s G \cup \{\Gamma, \phi \triangleleft \Delta\}$$

and by definition, we have

$$v + \omega^z \geq \rho(d) = \max\{|\phi| + 1, \rho(d_0), \rho(d_1)\}.$$

and, as the last rule was not a weakening, we get $|d_i| < |d|$. We can apply the induction hypothesis to d_0, d_1 to get derivations $d'_0 \vdash^s G \cup \{\Gamma \triangleleft \phi\}$ and $d'_1 \vdash^s G \cup \{\Gamma, \phi \triangleleft \Delta\}$ with $|d'_i| \leq \chi^z(|d_i|)$ and $\rho(d'_i) \leq v$.

If $z = 0$, then $\rho(d) \leq v + \omega^z = v + 1$ and therefore $|\phi| \leq v$. Now, either

- (i) $\max\{\rho(d'_0), \rho(d'_1)\} \leq |\phi|$, or
- (ii) $\max\{\rho(d'_0), \rho(d'_1)\} > |\phi|$.

For (i), we may apply Lemma 6.7 to get a derivation

$$d' \vdash G \cup \{\Gamma \triangleleft \Delta\}$$

with $\rho(d') \leq |\phi| \leq v$ and

$$\begin{aligned} |d'| &\leq 2 \otimes (\chi^0(|d_0|) \oplus \chi^0(|d_1|)) \\ &\leq 2 \otimes (4^{\otimes |d_0|} \oplus 4^{\otimes |d_1|}) \\ &\leq 2 \otimes (2 \otimes 4^{\otimes \max\{|d_0|, |d_1|\}}) \\ &\leq 4^{\otimes (\max\{|d_0|, |d_1|\} + 1)} \\ &= 4^{\otimes |d|} = \chi^0(|d|) \end{aligned}$$

where we in particular used that \oplus and \otimes are increasing in both arguments as well as commutative, associative and distributive, that χ^0 is increasing, that $\alpha \oplus 1 = \alpha + 1$, and that $\max\{|d_0|, |d_1|\} + 1 = |d|$ by definition.

For (ii), we combine d'_0 and d'_1 using cut on ϕ to a derivation $d' \vdash G \cup \{\Gamma \triangleleft \Delta\}$. Now, we get

$$\rho(d) = \max\{|\phi| + 1, \rho(d'_0), \rho(d'_1)\} = \max\{\rho(d'_0), \rho(d'_1)\} \leq v$$

as $|\phi| + 1 \leq \max\{\rho(d'_0), \rho(d'_1)\}$ by the assumption (ii) and as $\rho(d'_i) \leq v$ from the induction hypothesis. Further, we have

$$|d'| = \max\{|d'_0|, |d'_1|\} + 1 \leq \max\{4^{\otimes |d_0|}, 4^{\otimes |d_1|}\} + 1 \leq 4^{\otimes |d|} = \chi^0(|d|)$$

which completes the case for (ii).

If $z \neq 0$, then there are $y < z$ and $k \in \mathbb{N}$ such that $\rho(\phi) \leq v + \omega^y \cdot k$. We can combine d'_0 and d'_1 using cut to a derivation $\hat{d} \vdash^s G \cup \{\Gamma \triangleleft \Delta\}$.

This immediately gives $|\hat{d}| \leq \max\{\chi^z(|d_0|), \chi^z(|d_1|)\} + 1$ and $\rho(\hat{d}) \leq v + \omega^y \cdot k$. As $y < z$, we can apply the induction hypothesis k -times and get a derivation $d' \vdash^s G \cup \{\Gamma \triangleleft \Delta\}$ with $\rho(d') \leq v$ as well as

$$|d'| \leq (\chi^y)^{(k)}(\max\{\chi^z(|d_0|), \chi^z(|d_1|)\} + 1)$$

We have $|d_i| < |d|$ and thus $\chi^z(|d_i|) < \chi^z(|d|)$ which implies $\chi^z(|d_i|) + 1 \leq \chi^z(|d|)$. This gives $\max\{\chi^z(|d_0|), \chi^z(|d_1|)\} + 1 \leq \chi^z(|d|)$.

Now, assuming $\alpha \leq \chi^z(\beta)$ and $y < z$, then

$$\chi^y(\alpha) \leq \chi^y(\chi^z(\beta)) = \chi^z(\beta)$$

where the first inequality follows from χ^y being increasing and the latter equality follows from the definition of $\chi^z(\beta)$ being the β -th simultaneous solution for $\gamma = \chi^x(\gamma)$ for all $x < z$. This implies, in combination with the above, that

$$|d'| \leq (\chi^y)^{(k)}(\max\{\chi^z(|d_0|), \chi^z(|d_1|)\} + 1) \leq \chi^z(|d|).$$

If the last rule was a weakening, let d_0 be the direct predecessor with $d_0 \vdash^s H_0$. Then by definition $\rho(d_0) = \rho(d)$. Depending on whether the weakening was internal or external, we get $w(d_0) < w(d)$ or $W(d_0) < W(d)$, respectively. Thus, the induction hypothesis applies to d_0 and H_0 and we get $d'_0 \vdash^s H_0$ with $|d'_0| \leq \chi^z(|d_0|)$ and $\rho(d'_0) \leq v$. By applying the same weakening again, we get some derivation $d' \vdash^s H$ with $|d'| = |d'_0| \leq \chi^z(|d_0|) = \chi^z(|d|)$ and with $\rho(d') = \rho(d'_0) \leq v$. \square

From Lemma 6.8, cut-elimination immediately follows:

Theorem 6.9 (Cut-elimination). *For any derivation $d \vdash^s H$, there exists a cut-free derivation $d' \vdash^s H$.*

Further, we of course get a bound on $|d'|$ which, in the case of finitary proofs with finitary formulas, matches that of Baaz and Ciabattoni [4].

Remark 6.10. One could immediately introduce versions of the systems considered before which use the traditional hypersequents based on multisets. One can then generalize Lemma 2 from [4] to these systems and by that obtain a cut-elimination result for them.

7. THE RANGE OF THE RESULTS AND EXTENSIONS

We want to use this section to give an overview over some other topics extending the previous ones and some initial observations, at varying depth, regarding those.

7.1. Extensions of the Completeness Results. At first, it should be noted that the completeness theorems for \mathbf{G}_{ω_1} and $\mathbf{G}_{\omega_1, \omega}$ do not generalize to uncountable sets Γ . Consider the following two notions of compactness from [1]:

Definition 7.1 (essentially Aguilera [1]). Let $\Vdash \subseteq \mathcal{P}(\mathcal{L}_{\kappa, \lambda}) \times \mathcal{L}_{\kappa, \lambda}$ or $\Vdash \subseteq \mathcal{P}(\mathcal{L}_{\kappa}) \times \mathcal{L}_{\kappa}$ be a relation. Then \Vdash is called

- (1) *weakly compact* if for every Γ, ϕ with at most κ many different atomics, there is a Δ with size $< \kappa$ such that $\Gamma \Vdash \phi$ implies $\Delta \Vdash \phi$,
- (2) *compact* if for every Γ, ϕ , there is a Δ with size $< \kappa$ such that $\Gamma \Vdash \phi$ implies $\Delta \Vdash \phi$.

Indeed, the relations $\vdash_{\mathbf{G}_{\omega_1}^{(D)}}$ and $\vdash_{\mathbf{G}_{\omega_1, \omega}^{(D)}}$ are compact as any proof only involves countably many formulas.

However, as Aguilera shows in [1], the consequence relation $\models_{\mathbf{G}_{\omega_1, \omega}}$ (i.e. $\models_{\{0,1\}_{\mathbb{R}}}$) is not even weakly compact. This follows from the following general result:

Proposition 7.2 (Aguilera [1]). *If $\models_{\mathbf{G}_{\kappa, \omega}}$ is weakly compact, then κ is weakly compact, i.e. it is strongly inaccessible and any tree of size κ , such that every level has $< \kappa$ nodes, has a branch of length κ .*

As ω_1 is not strongly inaccessible, it is not weakly compact and therefore $\models_{\mathbf{G}_{\omega_1, \omega}}$ is not either.

It turns out that Aguilera's proof is, in itself, "propositional" and can be straightforwardly adapted to \mathbf{G}_{κ} . We therefore have the following result which implies the same limitation for \mathcal{G}_{κ} :

Proposition 7.3. *If $\models_{\mathbf{G}_\kappa}$ is weakly compact, then κ is weakly compact.*

We omit the proof as it is, as mentioned above, literally that of Aguilera [1], rephrased in the propositional language.

Before moving on to other topics, we want to mention a peculiar application of the above result of Aguilera and its propositional version. Although we don't dive into the rich (and difficult) topic of interpolation for Gödel logics, we can use the fact that $\vdash_{\mathcal{G}_{\omega_1}^{(D)}}$ and $\vdash_{\mathbf{G}_{\omega_1}^{(D)}}$ are compact while $\models_{\mathbf{G}_{\omega_1, \omega}}$ and $\models_{\mathbf{G}_{\omega_1}}$ are not even weakly compact to show the following negative result.

Proposition 7.4. *There is no countable $\Delta \subseteq \mathcal{L}_{\omega_1, \omega}$ such that for any $\phi, \psi \in \mathcal{L}_{\omega_1, \omega}$, $\models_{\mathbf{G}_{\omega_1, \omega}} \phi \rightarrow \psi$ implies that there is a $\delta \in \Delta$ with $\models_{\mathbf{G}_{\omega_1, \omega}} \phi \rightarrow \delta$ and $\models_{\mathbf{G}_{\omega_1, \omega}} \delta \rightarrow \psi$. Similarly for \mathcal{L}_{ω_1} and \mathbf{G}_{ω_1} .*

Proof. Suppose such a Δ would exist. Constructing the Lindenbaum-Tarski algebra over $\mathcal{L}_{\omega_1, \omega}$, the set $\{[\delta] \mid \delta \in \Delta\}$ would form a countable dense subset of \mathbf{LT}^Γ , i.e. for any $\phi, \psi \in \mathcal{L}_{\omega_1, \omega}$ with $[\phi] \leq^{\mathbf{LT}^\Gamma} [\psi]$ there exists a $\delta \in \Delta$ such that $[\phi] \leq^{\mathbf{LT}^\Gamma} [\delta] \leq^{\mathbf{LT}^\Gamma} [\psi]$, and that even for *uncountable* Γ . As \mathbf{LT}^Γ is therefore separable, it embeds into $[0, 1]_{\mathbb{R}}$ with an embedding preserving infima and suprema (by first embedding $\mathbf{LT}^\Gamma \cap \{[\delta] \mid \delta \in \Delta\}$ into $[0, 1]_{\mathbb{Q}}$ and then extending that embedding to \mathbf{LT}^Γ by completion). If we assume $\Gamma \not\vdash_{\mathcal{G}_{\omega_1, \omega}^D} \phi$ with said uncountable Γ , then this embedding would provide a countermodel by Lemma 4.14, verifying $\Gamma \not\models_{\mathbf{G}_{\omega_1, \omega}} \phi$. Thus, we would have completeness of $\mathcal{G}_{\omega_1, \omega}^D$ for $\mathbf{G}_{\omega_1, \omega}$ w.r.t. uncountable Γ which is a contradiction to Proposition 7.2.

The argument works similarly also for the propositional \mathbf{G}_{ω_1} . \square

Although it is probably expected that there is no countable set of interpolants in this infinitary case (as we already work over an uncountable language), it is maybe still instructive to note how cardinality considerations can have an impact on these type of questions.

In a similar vein, a generalization of the completeness results to $\mathcal{L}_{\omega_1, \omega_1}$ is problematic. As is well-known by a theorem of Scott (and Karp, see [23]²), the set of classical validities over $\mathcal{L}_{\omega_1, \omega_1}$ is not definable in $H(\omega_1)$, the collection of hereditarily countable sets, and thus in particular not Σ_1 on $H(\omega_1)$, a property which would, however, be implied by the existence of a complete classical proof calculus with proofs of countable lengths. These results should generalize to the Gödel case:

Question 7.5. *Is the set of theorems of $\mathbf{G}_{\omega_1, \omega_1}$ non-definable over $H(\omega_1)$?*

7.2. Other Sets of Variables. As common in propositional and first-order Gödel logics, one could consider closed sets V with $\{0, 1\} \subseteq V \subsetneq [0, 1]$ instead of $[0, 1]$ as truth-value sets, thereby forming the propositional variants \mathbf{G}_κ^V and the first-order variants $\mathbf{G}_{\kappa, \lambda}^V$ by extending the semantic definitions from before to take values in V . The most common instances of V are among

$$\begin{aligned} V_{\mathbb{R}} &:= [0, 1], \\ V_0 &:= \{0\} \cup [1/2, 1], \\ V_{\downarrow} &:= \{1/k \mid k \geq 1\} \cup \{0\}, \\ V_{\uparrow} &:= \{1 - 1/k \mid k \geq 1\} \cup \{1\}, \\ V_n &:= \{1 - 1/k \mid 1 \leq k \leq n - 1\} \cup \{1\} \text{ with } n \geq 2, \end{aligned}$$

following the selection from [8]. In that notation, we have $\mathbf{G}_\kappa = \mathbf{G}_\kappa^{V_{\mathbb{R}}}$ and $\mathbf{G}_{\kappa, \lambda} = \mathbf{G}_{\kappa, \lambda}^{V_{\mathbb{R}}}$. A few easy observations from [8] directly carry over to the infinitary case.

Proposition 7.6. *Let \mathbf{G}^V be either \mathbf{G}_κ^V or $\mathbf{G}_{\kappa, \lambda}^V$ for arbitrary κ, λ (for $\lambda \leq \kappa$). The following relations hold:*

- (1) $\mathbf{G}^{V_{\mathbb{R}}} = \bigcap_V \mathbf{G}^V$
- (2) $\mathbf{G}^{V_n} \supseteq \mathbf{G}^{V_{n+1}}$
- (3) $\mathbf{G}^{V_n} \supseteq \mathbf{G}^{V_{\uparrow}} \supseteq \mathbf{G}^{V_{\mathbb{R}}}$
- (4) $\mathbf{G}^{V_n} \supseteq \mathbf{G}^{V_{\downarrow}} \supseteq \mathbf{G}^{V_{\mathbb{R}}}$
- (5) $\mathbf{G}^{V_0} \supseteq \mathbf{G}^{V_{\mathbb{R}}}$.

We omit the proof as it is essentially a replica of the analogous result in the finitary first-order case (see [8]). Still, we want to emphasize the following differences to the finitary case: already in the finitary propositional case, $\mathbf{G}_\omega^{V_{\uparrow}}$, $\mathbf{G}_\omega^{V_{\downarrow}}$ and $\mathbf{G}_\omega^{V_0}$ differ in entailment (see [10]) while they coincide, as observed first by Dummett [15], in tautologies unlike the finitary first-order versions. So, it is natural that in this infinitary case the propositional (and first-order) versions differ in entailment as well.

²As Karp remarks, the proof of hers is based on an outline circulated by Scott in 1960 which was not published.

But, in the infinitary case, the propositional variants already differ with respect to tautologies and, moreover, the witnessing (non-)tautologies are natural analogues of the finitary first-order examples: consider $C^\uparrow := \bigvee_{i \in \omega} (p_i \rightarrow \bigwedge_{j \in \omega} p_j)$, $C^\downarrow := \bigvee_{i \in \omega} (\bigvee_{j \in \omega} p_j \rightarrow p_i)$ and $\text{ISO}_0 := \bigwedge_{i \in \omega} \neg \neg p_i \rightarrow \neg \neg \bigwedge_{i \in \omega} p_i$.

Then C^\uparrow is valid in G^{V^\uparrow} but not in G^{V^\downarrow} . C^\downarrow is valid in G^{V^\uparrow} and G^{V^\downarrow} . Both are not valid in G^{V_0} and $G^{V_{\mathbb{R}}}$. ISO_0 is valid in G^{V_0} but not in $G^{V_{\mathbb{R}}}$.

Further, we can give the following analogous result on the relationship between G^{V^\uparrow} and the finite-valued G^{V_n} .

Proposition 7.7. *We have*

$$G_{\kappa}^{V^\uparrow} = \bigcap_{n \geq 2} G_{\kappa}^{V_n} \quad \text{and} \quad G_{\kappa, \lambda}^{V^\uparrow} = \bigcap_{n \geq 2} G_{\kappa, \lambda}^{V_n}.$$

Proof. Again, let G^V be either G_{κ}^V or $G_{\kappa, \lambda}^V$. Item (3) of Proposition 7.6 gives

$$G^{V^\uparrow} \subseteq \bigcap_{n \geq 2} G^{V_n}.$$

For the converse, suppose that $\Gamma \not\models_{G^{V^\uparrow}} \phi$, i.e. there is an evaluation v such that $v[\Gamma] \subseteq \{1\}$ but $v(\phi) < 1$ in the propositional case or an interpretation \mathfrak{J} such that $\overline{\mathfrak{J}}[\Gamma] \subseteq \{1\}$ but $\overline{\mathfrak{J}}(\phi) < 1$ in the first-order case. As v (respectively $\overline{\mathfrak{J}}$) evaluates into V_{\uparrow} , there is a k such that $v(\phi) = 1 - \frac{1}{k}$ (respectively $\overline{\mathfrak{J}}(\phi) = 1 - \frac{1}{k}$). Let $x \in [0, 1]$ be such that $1 - \frac{1}{k} < x < 1 - \frac{1}{k+1}$ and

$$x \notin v[\text{sub}(\Gamma \cup \{\phi\})]$$

in the propositional case or such that

$$x \notin \text{Val}_{\mathfrak{J}}(\Gamma \cup \{\phi\})$$

in the first-order case. We form v_x or \mathfrak{J}_x by Lemma 4.9 or Lemma 4.15, respectively. The above choice of x is such that $\overline{v_x}[\Gamma] \subseteq \{1\}$ but $\overline{v_x}(\phi) < 1$ in the propositional case and $\overline{\mathfrak{J}_x}[\Gamma] \subseteq \{1\}$ but $\overline{\mathfrak{J}_x}(\phi) < 1$ in the first-order case by the previous lemmas. But, by the choice of x , we have that $\overline{v_x}$ or $\overline{\mathfrak{J}_x}$ evaluate into V_{k+1} which gives $\Gamma \not\models_{G^{V_{k+1}}} \phi$. \square

By the results of [8, 15], the status quo on complete proof calculi in the finitary setting is very clear cut: $G_{\omega, \omega}^V$ is axiomatizable iff V is finite or uncountable with either 0 contained in the perfect kernel of V or isolated. In particular, already the tautologies of $G_{\omega, \omega}^{V^\uparrow}$ and $G_{\omega, \omega}^{V^\downarrow}$ are not recursively enumerable. On the propositional side, while the tautologies of all G_{ω}^V are axiomatizable (again, see [15]), the only axiomatizable entailment relations are $G_{\omega}^{V_n}$ and $G_{\omega}^{V_{\mathbb{R}}}$ (see [10]).

Now, the situation is different in the infinitary cases. In the following, we will obtain analogous axiomatizations for the instances of V which were axiomatizable already in the finite but we further obtain infinitary axiomatizations of $G_{\omega_1, \omega}^{V^\uparrow}$ and $G_{\omega_1}^{V^\uparrow}$.

We don't know the state of $G_{\omega_1, \omega}^V / G_{\omega_1}^V$ for $V = V_{\downarrow}$ or any other V but in the finitary, as shown by Hájek [19], the tautologies of $G_{\omega, \omega}^{V^\downarrow}$ are not arithmetical. So we leave with the following question regarding the other truth-value sets:

Question 7.8. *Have any $G_{\omega_1}^V$ or $G_{\omega_1, \omega}^V$ (countable) infinitary axiomatizations for any V which has not been considered here?*

To approach the axiomatizability questions, we follow the general route of [8] which relies on tools from the theory of Polish spaces, like the Cantor-Bendixson theorem, which we briefly want to recall. In the following, we write \mathbf{A}_V for the Heyting algebra associated with a Gödel set V . Note that every V , as a closed subset of \mathbb{R} , is a Polish space. A subset P of \mathbb{R} is *perfect* if it is closed and every point is a limit point in the topology induced by \mathbb{R} .

Theorem 7.9 (Cantor-Bendixson). *Any Polish space X can be partitioned as $X = P \cup C$ with P perfect and C countable and open.*

The following result is then the central connection between the Cantor-Bendixson theorem and evaluations over Gödel sets.

Lemma 7.10 (Preining [28]). *Let $M \subseteq [0, 1]$ be countable and $P \subseteq [0, 1]$ be perfect. Then there is a strictly monotone $h : M \rightarrow P$ which preserves any infima and suprema existing in M and if $\inf M \in M$, then $h(\inf M) = \inf P$.*

7.2.1. *V is finite.* We consider the axiom scheme

$$\text{FIN}(n) := (\phi_0 \rightarrow \phi_1) \vee (\phi_1 \rightarrow \phi_2) \vee \cdots \vee (\phi_{n-1} \rightarrow \phi_n)$$

as in the finitary axiomatizations.

Theorem 7.11. *For any countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\omega_1}$, we have*

$$\Gamma \vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}(n)} \phi \text{ iff } \Gamma \models_{\mathbf{G}_{\omega_1}^{V_n}} \phi.$$

Similarly, for any countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\omega_1, \omega}$ where all formulas of Γ are closed, we have

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^D + \text{FIN}(n)} \phi \text{ iff } \Gamma \models_{\mathbf{G}_{\omega_1, \omega}^{V_n}} \phi.$$

In fact, $\text{FIN}(n)$ can, in both cases, be replaced by $\text{FIN}^a(n) := \text{“all atomic instances of } \text{FIN}(n)\text{”}$.

Proof. Soundness is routine. For the converse, define $\mathcal{L}_A := \text{frag}(\Gamma \cup \{\phi\})$ and write \mathcal{L}_A^a for the atomics of \mathcal{L}_A . We consider

$$\Pi := \{(\phi_0 \rightarrow \phi_1) \vee (\phi_1 \rightarrow \phi_2) \vee \cdots \vee (\phi_{n-1} \rightarrow \phi_n) \mid \phi_i \in \mathcal{L}_A^a\}.$$

Π is countable as Γ , and therefore \mathcal{L}_A , is countable. Now suppose $\Gamma \not\vdash_{(\mathcal{G}_{\omega_1}^D + \text{FIN}(n))(\mathcal{L}_A)} \phi$. Then clearly $\Gamma \cup \Pi \not\vdash_{\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)} \phi$ and by strong completeness of $\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)$, we have $\Gamma \cup \Pi \not\models_{[0,1]_{\mathbb{R}}(\mathcal{L}_A)} \phi$, i.e. there is an evaluation $v : \mathcal{L}_A \rightarrow [0, 1]$ with

$$v[\Gamma \cup \Pi] \subseteq \{1\} \text{ but } v(\phi) < 1.$$

Now, the set $v[\mathcal{L}_A^a]$ contains at most n elements as if not, then there are formulas $\phi_0, \dots, \phi_n \in \mathcal{L}_A^a$ with $v(\phi_i) > v(\phi_{i+1})$. In that case, we have

$$v((\phi_0 \rightarrow \phi_1) \vee (\phi_1 \rightarrow \phi_2) \vee \cdots \vee (\phi_{n-1} \rightarrow \phi_n)) < 1$$

which is a contradiction to $v[\Pi] \subseteq \{1\}$. Thus, we can write $v[\mathcal{L}_A^a] \subseteq \{0, v_1, \dots, v_{n-2}, 1\}$ with $v_i < v_{i+1}$. By induction on the structure of formulas, we also get $v[\mathcal{L}_A] \subseteq \{0, v_1, \dots, v_{n-2}, 1\}$. We define a function $h : v[\mathcal{L}_A] \rightarrow V_n$ by setting $h(0) := 0$, $h(1) := 1$ and $h(v_i) := 1 - \frac{1}{i+1}$. $v[\mathcal{L}_A]$ is, with its order $<$, a Heyting algebra and therefore h is an isomorphism of Heyting algebras and in particular preserves infima and suprema. Lemma 4.8 gives that $h \circ v$ is an \mathcal{L}_A -evaluation with

$$(h \circ v)[\Gamma] \subseteq \{1\} \text{ but } (h \circ v)(\phi) < 1.$$

As $h \circ v$ evaluates into V_n , we get $\Gamma \not\models_{\mathbf{A}_{V_n}(\mathcal{L}_A)} \phi$. Therefore also $\Gamma \not\models_{\mathbf{G}_{\omega_1}^{V_n}} \phi$, as \mathbf{A}_{V_n} is complete.

For the first-order instances, we instead consider

$$\Pi := \{\forall x_{i_1} \dots \forall x_{i_m} ((\phi_0 \rightarrow \phi_1) \vee \cdots \vee (\phi_{n-1} \rightarrow \phi_n)) \mid \phi_j \in \mathcal{L}_A^a, \text{var}(\phi_j) \subseteq \{x_{i_1}, \dots, x_{i_m}\}\}.$$

similar to [8] where \mathcal{L}_A^a now represents the atomic formulas of the first-order fragment \mathcal{L}_A .

For countable fragments \mathcal{L}_A , Π is countable and as every atomic formulas has only a finite number of variables, we have that every formula in Π is closed. Then apply Lemma 4.14 in place of Lemma 4.8 as in [8]. \square

7.2.2. *V is V_{\uparrow} .* The strength of infinitary logics is of course that we have infinitary disjunctions available which we can use to combine the various finitary axioms $\text{FIN}(n)$. More precisely, we define the scheme FIN by

$$\bigvee_{n \geq 2} \bigwedge_{k \in \omega} \bigvee_{i=0}^{n-1} (\phi_i^{n,k} \rightarrow \phi_{i+1}^{n,k}).$$

In the first-order case, we will additionally consider a seemingly weakened version FIN^a given by

$$\bigvee_{n \geq 2} \bigwedge_{k \in \omega} \bigvee_{i=0}^{n-1} \forall x_{j_1}^{n,k} \dots \forall x_{j_m}^{n,k} (\phi_i^{n,k} \rightarrow \phi_{i+1}^{n,k})$$

where all $\phi_i^{n,k}$ are atomic with $\text{var}(\phi_i^{n,k}) \subseteq \{x_{j_1}^{n,k}, \dots, x_{j_m}^{n,k}\}$. Now, FIN can be used to obtain an axiomatization of $\bigcap_{n \geq 2} \mathbf{G}_{\omega_1}^{V_n}$ or $\bigcap_{n \geq 2} \mathbf{G}_{\omega_1, \omega}^{V_n}$ for countable sets. In combination with the previous Proposition 7.7, we then obtain an axiomatization of V_{\uparrow} .

Theorem 7.12. *For any countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\omega_1}$, we have*

$$\Gamma \vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}} \phi \text{ iff } \Gamma \models_{\mathbf{G}_{\omega_1}^{V_{\uparrow}}} \phi.$$

Similarly, for countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\omega_1, \omega}$ where all formulas in Γ are closed, we have

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^D + \text{FIN}} \phi \text{ iff } \Gamma \models_{\mathbf{G}_{\omega_1, \omega}^{V_{\uparrow}}} \phi.$$

Proof. FIN is valid in V_{\uparrow} : write $\psi^{n,k} := \bigvee_{i=0}^{n-1} (\phi_i^{n,k} \rightarrow \phi_{i+1}^{n,k})$ and suppose $v : \mathcal{L}_{\omega_1} \rightarrow V_{\uparrow}$ is such that

$$v \left(\bigvee_{n \geq 2} \bigwedge_{k \in \omega} \psi^{n,k} \right) < 1.$$

Thus, for some α , we have

$$v \left(\bigwedge_{k \in \omega} \psi^{n,k} \right) \leq \alpha < 1$$

for any $n \geq 2$. Let k be such that $v(\psi^{n,k}) < 1$. Such a k exists as $\alpha < 1$. For such a k , we get

$$v(\phi_0^{n,k}) > v(\phi_1^{n,k}) > \dots > v(\phi_n^{n,k})$$

and as v evaluates into V_{\uparrow} , we have

$$v(\phi_0^{n,k}) \geq 1 - \frac{1}{n+1} \text{ and } v(\phi_1^{n,k}) \geq 1 - \frac{1}{n}$$

and therefore $v(\phi_0^{n,k} \rightarrow \phi_1^{n,k}) \geq 1 - \frac{1}{n}$. This yields $v(\psi^{n,k}) \geq 1 - \frac{1}{n}$ for any such k and we get

$$\begin{aligned} v \left(\bigwedge_{k \in \omega} \psi^{n,k} \right) &= \min \left\{ \inf_{v(\psi^{n,k}) < 1} v(\psi^{n,k}), \inf_{v(\psi^{n,k}) = 1} v(\psi^{n,k}) \right\} \\ &= \inf_{v(\psi^{n,k}) < 1} v(\psi^{n,k}) \\ &\geq 1 - \frac{1}{n} \end{aligned}$$

for any $n \geq 2$. But this implies

$$v \left(\bigvee_{n \geq 2} \bigwedge_{k \in \omega} \psi^{n,k} \right) = 1$$

in contradiction to our assumption.

For the converse, suppose $\Gamma \not\vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}} \phi$. Then, we have

$$\Gamma \not\vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}(n)} \phi$$

for some n as if $\Gamma \vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}(n)} \phi$ for all n , then there are countably many instances

$$\bigvee_{i=0}^{n-1} (\phi_i^{n,k} \rightarrow \phi_{i+1}^{n,k}) \quad (k \in \omega)$$

of $\text{FIN}(n)$, such that

$$\Gamma \vdash_{\mathcal{G}_{\omega_1}^D} \bigwedge_{k \in \omega} \bigvee_{i=0}^{n-1} (\phi_i^{n,k} \rightarrow \phi_{i+1}^{n,k}) \rightarrow \phi$$

for any n and thus

$$\Gamma \vdash_{\mathcal{G}_{\omega_1}^D} \bigvee_{n \geq 2} \bigwedge_{k \in \omega} \bigvee_{i=0}^{n-1} (\phi_i^{n,k} \rightarrow \phi_{i+1}^{n,k}) \rightarrow \phi$$

by $(R\omega)_1$. The premise is an instance of FIN which implies that $\Gamma \vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}} \phi$, a contradiction. Therefore, there is an n with $\Gamma \not\vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}(n)} \phi$ and thus, using that Γ is countable, we get

$$(\Gamma, \phi) \notin \bigcap_{n \geq 2} \mathbf{G}_{\omega_1}^{V_n}$$

by Theorem 7.11 which implies $\Gamma \not\vdash_{\mathbf{G}_{\omega_1}^{V_{\uparrow}}} \phi$ by Proposition 7.7.

The first order case is very similar. Soundness follows in the same way and the converse follows by the following slightly modified argument: supposing $\Gamma \not\vdash_{\mathcal{G}_{\omega_1}^D + \text{FIN}} \phi$, we of course also have $\Gamma \not\vdash_{\mathcal{G}_{\omega_1, \omega}^D + \text{FIN}^a} \phi$. As in the propositional case, we get $\Gamma \not\vdash_{\mathcal{G}_{\omega_1, \omega}^D + \text{FIN}^a(n)} \phi$ for some n where it is essential that all instances of FIN^a are closed to be able to use the deduction theorem. We get $\Gamma \not\vdash_{\mathbf{G}_{\omega_1, \omega}^{V_{\uparrow}}} \phi$ using Proposition 7.7. \square

7.2.3. *0 contained in the perfect kernel.* We obtain the following infinitary version of the well-known finitary result from [8].

Lemma 7.13. *Let V be a Gödel set and P its perfect kernel and $W = V \cup [\inf P, 1]$. For any fragment \mathcal{L}_A (of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$) and any countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$, we have*

$$\Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_W(\mathcal{L}_A)} \phi.$$

Proof. We only present the propositional case which is essentially contained in [8]. The first order case is the same with Lemma 4.9 replaced by Lemma 4.15 and Lemma 4.8 replaced by Lemma 4.14, respectively

Since $V \subseteq W$, we have that $\Gamma \models_{\mathbf{A}_W(\mathcal{L}_A)} \phi$ implies $\Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi$.

Let $v : \mathcal{L}_A \rightarrow \mathbf{A}_W$ be such that $v[\Gamma] \subseteq \{1\}$ but $v(\phi) < 1$. As Γ is countable, there is an $x \in [0, 1]$ with $v(\phi) < x < 1$ and where $x \notin v[\text{sub}(\Gamma \cup \{\phi\})]$. With v_x as in Lemma 4.9, we have

$$\overline{v_x}(\psi) = \begin{cases} v(\psi) & \text{if } v(\psi) < x, \\ 1 & \text{otherwise,} \end{cases}$$

for any $\psi \in \Gamma \cup \{\phi\}$. Set $M := \{\overline{v_x}(\psi) \mid \psi \in \mathcal{L}_A\} \cup \{1\}$ and $M_1 := (M \cap [\inf P, x]) \cup \{\inf P\}$. Lemma 7.10 gives a strictly monotone $h : M_1 \rightarrow P$ preserving all infima and suprema such that $h(\inf M_1) = \inf P$. We define

$$g : y \mapsto \begin{cases} y & \text{if } y \in [0, \inf P], \\ h(y) & \text{if } y \in [\inf P, x], \\ 1 & \text{if } y = 1, \end{cases}$$

for $y \in M$. g preserves infima, suprema and is strictly monotone with $g(0) = 0$ and $g(1) = 1$. g therefore is a homomorphism of Heyting algebras preserving infima and suprema and by Lemma 4.8, $g \circ \overline{v_x}$ is an evaluation. But $g \circ \overline{v_x}$ maps into V and $(g \circ \overline{v_x})[\Gamma] \subseteq \{1\}$ but $g(\overline{v_x}(\phi)) = g(v(\phi)) < 1$. \square

This immediately yields the following completeness result.

Theorem 7.14. *Let V be a Gödel set where 0 is contained in the perfect kernel. For any fragment \mathcal{L}_A (of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$) and countable $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$ where Γ is closed in the first-order case, we have*

$$\Gamma \vdash_{\mathcal{G}^D(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi.$$

Here, we write $\mathcal{G}^D(\mathcal{L}_A)$ for $\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)$ or $\mathcal{G}_{\omega_1, \omega}^D(\mathcal{L}_A)$, respectively, depending the choice of language.

7.2.4. *V uncountable and 0 isolated.* We now turn to the case of an isolated 0 . For the first-order case, we have to additionally consider the quantifier version QISO_0 of ISO_0 (from which it was derived) as e.g. seen in [8]:

$$\forall x \neg \neg \phi \rightarrow \neg \neg \forall x \phi.$$

The following lemma is an easy adaption of the finitary first-order case from [8], which only mentions the latter statement regarding the quantifiers.

Lemma 7.15. *For any ϕ_i , we have $\vdash_{\mathcal{G}_{\omega_1}^D + \text{ISO}_0} \neg \bigwedge_{i \in \omega} \phi_i \rightarrow \bigvee_{i \in \omega} \neg \phi_i$. Similarly for $\mathcal{G}_{\omega_1, \omega}^D$. In the first-order case, we additionally have $\vdash_{\mathcal{G}_{\omega_1, \omega}^D + \text{QISO}_0} \neg \forall x \phi \rightarrow \exists x \neg \phi$ for any ϕ and x .*

Theorem 7.16. *Let V be an uncountable Gödel set with 0 isolated. Let \mathcal{L}_A be a countable fragment of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$ where $\bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A$ iff $\bigvee_{i \in \omega} \phi_i \in \mathcal{L}_A$ and where $\bigvee_{i \in \omega} \phi_i \in \mathcal{L}_A$ implies $\bigvee_{i \in \omega} \neg \phi_i \in \mathcal{L}_A$.*

For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$, we have

$$\Gamma \vdash_{(\mathcal{G}_{\omega_1}^D + \text{ISO}_0)(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi$$

in the propositional case and

$$\Gamma \vdash_{(\mathcal{G}_{\omega_1, \omega}^D + \text{ISO}_0 + \text{QISO}_0)(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi$$

in the first-order case where Γ is assumed to be closed.

Proof. Soundness is clear. For the other direction, note that we know by Lemma 7.13 that

$$\Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_{V \cup [\inf P, 1]}(\mathcal{L}_A)} \phi$$

where P is the perfect kernel of V . So w.l.o.g. $[\inf P, 1] \subseteq V$. We define

$$\Pi := \left\{ \neg \bigwedge_{i \in \omega} \phi_i \rightarrow \bigvee_{i \in \omega} \neg \phi_i \mid \bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A \right\}.$$

By the assumptions on \mathcal{L}_A , we have $\Pi \subseteq \mathcal{L}_A$ and in particular, Π is countable. Suppose that $\Gamma \not\vdash_{(\mathcal{G}_{\omega_1}^D + \text{ISO}_0)(\mathcal{L}_A)} \phi$.

We now either have $\Pi \cup \Gamma \models_{[0,1]_{\mathbb{R}}(\mathcal{L}_A)} \phi$ or $\Pi \cup \Gamma \not\models_{[0,1]_{\mathbb{R}}(\mathcal{L}_A)} \phi$. For the former, we get

$$\Pi \cup \Gamma \vdash_{\mathcal{G}_{\omega_1}^D(\mathcal{L}_A)} \phi$$

by completeness. As by Lemma 7.15, $(\mathcal{G}_{\omega_1}^D + \text{ISO}_0)(\mathcal{L}_A)$ proves every element of Π , we have

$$\Gamma \vdash_{(\mathcal{G}_{\omega_1}^D + \text{ISO}_0)(\mathcal{L}_A)} \phi$$

which is a contradiction to our assumption. Therefore we have $\Pi \cup \Gamma \not\models_{[0,1]_{\mathbb{R}}(\mathcal{L}_A)} \phi$, i.e. there is a $v : \mathcal{L}_A \rightarrow [0, 1]$ with $v[\Pi] \cup v[\Gamma] \subseteq \{1\}$ and $v(\phi) < 1$. We define

$$h : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \inf P + \frac{x}{1 - \inf P} & \text{otherwise.} \end{cases}$$

Note that $(h \circ v)[\mathcal{L}_A] \subseteq V$. Uniquely extend $v_h : x \mapsto h(v(x))$ for $x \in \text{Var}_{\omega_1} \cap \mathcal{L}_A$ to \bar{v}_h on \mathcal{L}_A . Then we have

$$\bar{v}_h(\psi) = h(v(\psi))$$

for any $\psi \in \mathcal{L}_A$ which gives the claim. This can be proven by induction on ψ , see in particular [8] for the similar finitary case, where $v[\Pi] \subseteq \{1\}$ is used to handle the \wedge -case.

The proof in the first-order case is very similar. We then consider

$$\begin{aligned} \Pi := & \left\{ \forall \bar{x} \left(\neg \bigwedge_{i \in \omega} \phi_i \rightarrow \bigvee_{i \in \omega} \neg \phi_i \right) \mid \bigwedge_{i \in \omega} \phi_i \in \mathcal{L}_A, \bar{x} \in (\text{Var}_A)^n, n \in \mathbb{N} \right\} \\ & \cup \{ \forall \bar{y} (\neg \forall x \phi \rightarrow \exists x \neg \phi) \mid \phi \in \mathcal{L}_A, x \in \text{Var}_A, \bar{y} \in (\text{Var}_A)^n, n \in \mathbb{N} \} \end{aligned}$$

and one proceeds as above and obtains a \mathfrak{J} with $\bar{\mathfrak{J}}[\Pi] \cup \bar{\mathfrak{J}}[\Gamma] \subseteq \{1\}$ but $\bar{\mathfrak{J}}(\phi) < 1$. As Π is not closed, note in particular Remark 5.6 in that closedness is not needed for both directions of the completeness results, only for the soundness direction. Note that Π is countable.

With the same h defined as before, one similarly defines \mathfrak{J}'_h for $\mathfrak{J}' = \bar{\mathfrak{J}}_{\frac{m}{x}}$ by changing $P^{\mathfrak{M}}$ to $h \circ P^{\mathfrak{M}}$ for predicates P in the underlying model \mathfrak{M} and the key point is to now establish

$$\bar{\mathfrak{J}}'_h(\psi) = h(\bar{\mathfrak{J}}'(\psi))$$

for any $\psi \in \mathcal{L}_A$ for any such \mathfrak{J}' where, for the \wedge - and \forall -cases, it is important that $\bar{\mathfrak{J}}[\Pi] \subseteq \{1\}$ and that any element of Π is universally quantified with arbitrary but finitely many quantifiers such that $\bar{\mathfrak{J}}[\Pi] \subseteq \{1\}$ implies

$$\bar{\mathfrak{J}}' \left(\neg \bigwedge_{i \in \omega} \phi_i \rightarrow \bigvee_{i \in \omega} \neg \phi_i \right) = 1$$

and

$$\bar{\mathfrak{J}}'(\neg \forall x \phi \rightarrow \exists x \neg \phi) = 1$$

even for any \mathfrak{J}' as above. □

As before, we can now lift the above result to arbitrary \mathcal{L}_A if we restrict to countably many assumptions.

Corollary 7.17. Let V be an uncountable Gödel set with 0 isolated and let \mathcal{L}_A be an arbitrary fragment of \mathcal{L}_{ω_1} or $\mathcal{L}_{\omega_1, \omega}$ with the same additional closure conditions as in Theorem 7.16 but let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$ be countable with Γ closed in the first-order case.

For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_A$, we have

$$\Gamma \vdash_{(\mathcal{G}_{\omega_1}^D + \text{ISO}_0)(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi$$

in the propositional case and

$$\Gamma \vdash_{(\mathcal{G}_{\omega_1, \omega}^D + \text{ISO}_0 + \text{QISO}_0)(\mathcal{L}_A)} \phi \text{ iff } \Gamma \models_{\mathbf{A}_V(\mathcal{L}_A)} \phi$$

in the first-order case.

The proof follows the same type of argument as in Corollary 5.5 where we now consider the smallest fragment containing $\Gamma \cup \{\phi\}$ with the additional closure properties. The important point is here that for a countable Γ , this fragment is again countable.

Again, the above gives in particular

$$\Gamma \vdash_{\mathcal{G}_{\omega_1}^D + \text{ISO}_0} \phi \text{ iff } \Gamma \models_{\mathbf{C}_{V_1}^V} \phi$$

for countable Γ and

$$\Gamma \vdash_{\mathcal{G}_{\omega_1, \omega}^D + \text{ISO}_0 + \text{QISO}_0} \phi \text{ iff } \Gamma \models_{\mathbf{C}_{V_1, \omega}^V} \phi$$

for countable and closed Γ where 0 is isolated in V .

Acknowledgments. I want to thank Matthias Baaz for helpful discussions on the topics of this paper. Further, I want to thank the anonymous referees for their helpful comments and interesting suggestions, in particular for pointing out the reference [26].

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