GÖDEL JUSTIFICATION LOGICS AND REALIZATION

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Abstract. We study the topic of realization from classical justification logics in the context of the recently introduced Gödel justification logics. We show that the standard Gödel modal logics of Caicedo and Rodriguez are not realized by the Gödel justification logics and followingly, we study possible extensions of the Gödel justification logics which are strong enough to realize the standard Gödel modal logics. On the other hand, we study the fragments of the standard Gödel modal logics which are realized by the usual Gödel justification logics. We prove the corresponding Realization Theorem by using Fitting's *merging of realizations* as well as appropriate hypersequent calculi on the modal side, adapting the work of Metcalfe and Olivetti. For these hypersequent calculi, we also show a cut-elimination theorem. We provide natural semantical characterizations for all of these newly introduced logics.

1. Introduction

The study of justification logics began in the '90s when Artemov introduced the *logic of proofs* (see [1, 2]) to provide an arithmetical interpretation of the classical modal logic $\mathcal{S}4$ in Peano arithmetic, a possibility anticipated by Gödel in [18] (where he actually introduced the modern Hilbert-style calculus for $\mathcal{S}4$). In comparison to the usual necessity statements "$\Box\phi$" of unexplicit modal logics, the logic of proofs contains modal formulae of the form "$t : \phi$", indexed by so-called *proof terms* t which encode an explicit reason for the necessity statement directly in the language. In that way, one can also reason about the dynamics *within* these reasons of necessity statements in addition to just reasoning about dynamics *between* necessity statements. Artemov provided a complete arithmetical semantics for this logic of proofs, where proof terms are interpreted by codes of proofs in classical Peano arithmetic, and further embedded $\mathcal{S}4$ into the logic of proofs which, in combination, yielded the arithmetical completeness result of $\mathcal{S}4$. This embedding is established by the so-called *Realization Theorem*: for every modal theorem of $\mathcal{S}4$, the different occurrences of $\Box$'s in the theorem can be replaced by appropriate proof terms to yield a theorem of the logic of proofs.

In the following years, research into the logic of proofs yielded various subsystems and extensions forming the modern framework of justification logics (see [3] for a survey and the recent textbooks [4, 25]) and the development of a corresponding epistemic semantics, spanning this whole framework, prompted that the interpretation of the proof terms broadened to representing general epistemic justifications, lifting the whole framework into formal epistemology. While the arithmetical semantics remains a feature confined pretty much to the logic of proofs, the Realization Theorem extends to these other justification logics together with corresponding, *unexplicit*, modal companions and forms the central relationship between justification logics and unexplicit modal logics.

In this paper, we investigate this property for fuzzy variants of modal and justification logics, namely for the standard Gödel modal logics as introduced by Caicedo and Rodriguez in [9] and the Gödel justification logics as introduced by Ghari in [15] and by Pischke in [29]. These variants replace the typical boolean base of classical justification (or modal) logics with $[0, 1]$-valued Gödel logic, one of the three main t-norm based fuzzy logics in the sense of Hájek [19] (although it initially originated from an intuitionistic perspective along the lines of Gödel [17], Dummett [11] and Horn [20]).

On the modal side, a first resulting difference to the classical case is that the natural semantic dual of $\Box$ in the context of the standard Gödel modal logics is not internally definable while the natural semantic dual operator of $\Box$ in classical modal logics is (via $\Diamond\theta := \neg\Box\neg\theta$). This gives rise to three different types of fuzzy Gödel modal logics: bi-modal versions containing both $\Box$- and $\Diamond$ as primitives interpreted by their respective semantic duals (see [10]) and the respective $\Box$ and $\Diamond$-fragments (see [9]).

As the justification modality "t :", in its standard (semantical) interpretation (both classically and in the Gödel-case), is a necessity-style operator, we only consider the $\Box$-fragments of the standard Gödel modal logics as there is no immediate dual notion of "t :", neither in the context of the classical nor in the context of the Gödel justification logics.[1] Therefore, there is also no immediate way of interpreting $\Diamond$ in the fuzzy justification setting.

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[1]There is some work on providing explicit analogues of the $\Diamond$-operator in some settings where it represents a (semantic) dual which is not internally definable, especially [23] in the context of *constructive modal logics*.

In this setup, we find that the standard Gödel modal logics are not realized by the Gödel justification logics. This gives rise to two natural questions. For one: what are the fragments of the standard Gödel modal logics which are realized by the standard Gödel justification logics? For another: what are feasible extensions of the standard Gödel justification logics which do realize the standard Gödel modal logics? We study both of these questions in this paper.

For the latter, we introduce extensions of the standard Gödel justification logics, defined over a broadened language augmented with a new operator on the justification terms, and show that adding a characteristic axiom for this operator to the previous axiomatic systems suffices to prove an analogue of the classical Realization Theorem in the many-valued cases with the usual versions of the standard Gödel modal logics. We also provide strongly complete semantics for these extensions based on the Gödel-Fitting and Gödel-Mkrtychev models previously introduced for the usual Gödel justification logics (see [29]).

For the former, we introduce fragments of the standard Gödel modal logics by dropping a specific problematic axiom in the usual Hilbert-style formulation and show that these fragments are the ones realized by the standard Gödel justification logics. We also devise a semantics for this realized fragment of the standard Gödel modal logics which suitably generalizes the $[0,1]$-valued semantics over Gödel-Kripke models from [9].

Both realization results are proved constructively using (partly new) hypersequent calculi for which we also provide a cut-elimination theorem. For this proof of the realization results, we adapt Fitting's merging of realizations (see [14]) to the Gödel-case.

2. Preliminaries

Throughout the paper, we write $x \odot y := \min\{x, y\}$ and $x \oplus y := \max\{x, y\}$ for $x, y \in [0,1]$.

2.1. Gödel justification logics.

Syntactically, we define the set of justification terms Jt by

$$Jt : t ::= c \mid x \mid [t + t] \mid [t \cdot t] \mid !t \mid ?t$$

with $c \in C := \{c_i \mid i \in \mathbb{N}\}$ (called a justification constant) and $x \in V := \{x_i \mid i \in \mathbb{N}\}$ (called a justification variable). The justification constants and variables are used to represent atomic justifications. Together with the operations $+, \cdot, !$ and $?$, the resulting terms are then able to model the dynamics of justifications under various styles of inference.

The corresponding language of justification logics $\mathcal{L}_J$ is then given by

$$\mathcal{L}_J : \phi ::= \bot \mid \top \mid p \mid (\phi \to \phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid t : \phi$$

with $t \in Jt$ and $p \in Var := \{p_i \mid i \in \mathbb{N}\}$ (called a propositional variable). Negation $\neg$ is introduced as a syntactical abbreviation via $\neg\phi := (\phi \to \bot)$. We write $\mathrm{var}(\phi)$ for the set of propositional variables occurring in a formula ϕ and $\mathrm{jvar}(t), \mathrm{jvar}(\phi)$ for the sets of justification variables occurring in some term t or some formula ϕ, respectively. A term t is called *closed* if it contains no justification variables, i.e. if $\mathrm{jvar}(t) = \emptyset$. Also, we write $\mathrm{sf}(\phi)$ for the set of all subformulae of ϕ (including ϕ).

2.1.1. *Proof calculi.* We define the following proof systems for the Gödel justification logics over $\mathcal{L}_J$, based on Hájek's strongly complete Hilbert-style proof calculus for propositional $[0,1]$-valued Gödel logic given in [19]:[2]

Definition 1. The Hilbert-style calculus $\mathcal{GJ}_0$ is given by the following axiom schemes and rules over $\mathcal{L}_J$:
($A1$): $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$;
($A2$): $(\phi \wedge \psi) \to \phi$;
($A3$): $(\phi \wedge \psi) \to (\psi \wedge \phi)$;
($A5a$): $(\phi \to (\psi \to \chi)) \to ((\phi \wedge \psi) \to \chi)$;
($A5b$): $((\phi \wedge \psi) \to \chi) \to (\phi \to (\psi \to \chi))$;
($A6$): $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$;
($A7$): $\bot \to \phi$;
($G4$): $\phi \to (\phi \wedge \phi)$;
($\top$): $\top \leftrightarrow \neg\bot$;
($\vee$): $(\phi \vee \psi) \leftrightarrow ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$;
(J): $t : (\phi \to \psi) \to (s : \phi \to [t \cdot s] : \psi)$;
($+$): $t : \phi \to [t + s] : \phi$, $s : \phi \to [t + s] : \phi$;
(MP): from $\phi \to \psi$ and ϕ, infer ψ.

By $\mathcal{G}$, we denote the fragment without the axiom schemes (J) and $(+)$. We then define the following axiomatic extensions of $\mathcal{GJ}_0$ over $\mathcal{L}_J$:

[2] In Hájek's work [19], the symbols $\top$ and $\vee$ are not primitives but introduced as abbreviations. The additional axiom schemes $(\top)$ and $(\vee)$ encode these definitions.

(1) $\mathcal{GJT}_0$ is the extension of $\mathcal{GJ}_0$ by the scheme $(F) : t : \phi \to \phi$;
(2) $\mathcal{GJ}4_0$ is the extension of $\mathcal{GJ}_0$ by the scheme $(!) : t : \phi \to !t : t : \phi$;
(3) $\mathcal{GLP}_0$ is the extension of $\mathcal{GJ}_0$ by the schemes (F) and $(!)$;
(4) $\mathcal{GJ}45_0$ is the extension of $\mathcal{GJ}_0$ by the schemes $(!)$ and $(?) : \neg t : \phi \to ?t : \neg t : \phi$;
(5) $\mathcal{GJT}45_0$ is the extension of $\mathcal{GJ}_0$ by the scheme (F), $(!)$ and $(?)$.

2.1.2. *Constant Specifications.* Let $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$. A *constant specification for* $\mathcal{GJL}_0$ is a set CS of formulae of the form

$$c_{i_n} : \cdots : c_{i_1} : \phi,$$

with $n \geq 1$ and where ϕ is an axiom instance of $\mathcal{GJL}_0$, where $c_{i_k} \in C$ whenever $1 \leq k \leq n$, and which is *downwards closed*, that is

$$\text{if } c_{i_n} : \cdots : c_{i_1} : \phi \in CS, \text{ then } c_{i_k} : \cdots : c_{i_1} : \phi \in CS$$

for every $k \in \{1, \dots, n\}$.

CS is called *axiomatically appropriate for* $\mathcal{GJL}_0$ if for every axiom instance ϕ of $\mathcal{GJL}_0$, there is a constant $c \in C$ such that $c : \phi \in CS$ and if it is *upwards closed*, that is

$$\text{if } c_{i_n} : \cdots : c_{i_1} : \phi \in CS, \text{ then } c_{i_{n+1}} : c_{i_n} : \cdots : c_{i_1} : \phi \in CS$$

for some $c_{i_{n+1}} \in C$.

CS is called *schematic for* $\mathcal{GJL}_0$ if, whenever ϕ and ψ are instances of the same axiom scheme of $\mathcal{GJL}_0$, then

$$c : \phi \in CS \text{ if, and only if } c : \psi \in CS$$

for any $c \in C$.

Naturally, there is only one *total* (that is, maximal with respect to $\subseteq$) constant specification CS for $\mathcal{GJL}_0$ defined by

$$c_{i_n} : \cdots : c_{i_1} : \phi \in CS$$

for every $n \geq 1$, every $i_1, \dots, i_n \in \mathbb{N}$ and every axiom instance ϕ of $\mathcal{GJL}_0$.

Given a constant specification CS for $\mathcal{GJL}_0$, we define the logic $\mathcal{GJL}_{CS}$ as the extension of $\mathcal{GJL}_0$ by the corresponding rule

$$(CS) : \text{from } c : \phi \in CS, \text{ infer } c : \phi.$$

Provability (under assumptions $\Gamma \subseteq \mathcal{L}_J$) of a formula $\phi \in \mathcal{L}_J$ in $\mathcal{GJL}_{CS}$ is defined as it is usually done in Hilbert-style calculi and is denoted by $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$.

A distinctive feature of $[0,1]$-valued propositional Gödel logic, in comparison with other many-valued logics based on t-norms, is that the classical Deduction Theorem holds for Gödel logic, that is the Gödel implication captures the deducibility relation. This carries over to the justification variants:

Lemma 1 (Deduction Theorem)**.** *Let*

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$$

and CS *be a constant specification for* $\mathcal{GJL}_0$. *For any* $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_J$, *we have*

$$\Gamma \cup \{\psi\} \vdash_{\mathcal{GJL}_{CS}} \phi \text{ iff } \Gamma \vdash_{\mathcal{GJL}_{CS}} \psi \to \phi.$$

The proof is a natural generalization of the classical case (see e.g. [25] for the case of $\mathcal{LP}$).

Another important result on the Gödel-based systems is the corresponding version of the *Lifting Lemma*, analogous to the classical case.

Lemma 2 (Lifting Lemma)**.** *Let*

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$$

and CS *be an axiomatically appropriate constant specification for* $\mathcal{GJL}_0$. *If* $\{\psi_1, \dots, \psi_n\} \vdash_{\mathcal{GJL}_{CS}} \phi$, *then for any justification terms* $t_1, \dots, t_n \in Jt$, *there is a justification term* $t \in Jt$ *such that*

$$\{t_1 : \psi_1, \dots, t_n : \psi_n\} \vdash_{\mathcal{GJL}_{CS}} t : \phi.$$

Also here, the proof is a straightforward generalization of the classical case (see e.g. [4]). Note that the condition $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathcal{GJL}_{CS}} \phi$ of the Lifting Lemma is equivalent to $\vdash_{\mathcal{GJL}_{CS}} \bigwedge_{i=1}^n \psi_i \to \phi$ by repeated applications of the Deduction Theorem and axiom scheme $(A5a)$.[3]

A direct consequence of the Lifting Lemma is the *Internalization Property* for Gödel justification logics with an axiomatically appropriate constant specification.

Corollary 1 (Internalization). Let

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$$

and CS be an axiomatically appropriate constant specification for $\mathcal{GJL}_0$. If $\vdash_{\mathcal{GJL}_{CS}} \phi$, then there is a $t \in Jt$ such that $\vdash_{\mathcal{GJL}_{CS}} t : \phi$.

Remark 2. By an examination of the usual proof of the above Lifting Lemma (see again [4] for the classical case), it can be seen that one can choose the term t such that the justification variables of t are among those of $t_1, \ldots, t_n$. In particular, in the above Internalization Property, t can be chosen to be a closed term.

2.1.3. *Semantics.* We recap the two main semantics for the Gödel justification logics. These many-valued models, generalizing the fundamental semantics of Fitting and Mkrtychev for the classical justification logics, were introduced in [16, 29].

We begin with the so-called Gödel-Mkrtychev models, many-valued analogues of the classical Mkrtychev models. The classical versions were originally introduced by Mkrtychev in [28] for the logic of proofs, by Kuznets in [21] for some of the other classical justification logics below the logic of proofs and in [24, 32] for the justification logics containing negative introspection.

Definition 2. A *Gödel-Mkrtychev model* is a structure $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ where

(1) $\mathcal{E} : Jt \times \mathcal{L}_J \to [0, 1]$,
(2) $e : Var \to [0, 1]$,

and which satisfies

(i) $\mathcal{E}(t, \phi \to \psi) \odot \mathcal{E}(s, \phi) \leq \mathcal{E}(t \cdot s, \psi)$,
(ii) $\mathcal{E}(t, \phi) \oplus \mathcal{E}(s, \phi) \leq \mathcal{E}(t + s, \phi)$,

for all $t, s \in Jt$ and all $\phi, \psi \in \mathcal{L}_J$.

We denote the class of all Gödel-Mkrtychev models by GM and say that $\mathfrak{M}$ is a GM-model if $\mathfrak{M} \in \mathsf{GM}$. We call a GM-model $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ *crisp* if both $\mathcal{E}$ and e only take values in $\{0, 1\}$.

For a GM-model $\mathfrak{M} = \langle \mathcal{E}, e \rangle$, we define its evaluation function $|\cdot|_{\mathfrak{M}} : \mathcal{L}_J \to [0, 1]$ by recursion over $\mathcal{L}_J$:

- $|\bot|_{\mathfrak{M}} := 0$; $|\top|_{\mathfrak{M}} := 1$;
- $|p|_{\mathfrak{M}} := e(p)$ for $p \in Var$;
- $|\phi \to \psi|_{\mathfrak{M}} := |\phi|_{\mathfrak{M}} \Rightarrow |\psi|_{\mathfrak{M}}$;
- $|\phi \wedge \psi|_{\mathfrak{M}} := |\phi|_{\mathfrak{M}} \odot |\psi|_{\mathfrak{M}}$;
- $|\phi \vee \psi|_{\mathfrak{M}} := |\phi|_{\mathfrak{M}} \oplus |\psi|_{\mathfrak{M}}$;
- $|t : \phi|_{\mathfrak{M}} := \mathcal{E}(t, \phi)$.

Here, and in the following, we write $\Rightarrow$ for the *residuum* of the minimum t-norm $\odot$, that is

$$x \Rightarrow y := \begin{cases} y & \text{if } x > y, \\ 1 & \text{otherwise,} \end{cases}$$

for $x, y \in [0, 1]$. For the derived connective $\neg$, we obtain the following derived truth function $\sim$:

$$\sim x := \begin{cases} 0 & \text{if } x > 0; \\ 1 & \text{otherwise.} \end{cases}$$

We also write $\sim^2 x$ for $\sim\sim x$.

We may extend evaluations to sets of formulae $\Gamma \subseteq \mathcal{L}_J$ by setting $|\Gamma|_{\mathfrak{M}} := \inf_{\phi \in \Gamma} |\phi|_{\mathfrak{M}}$. We write $\mathfrak{M} \models \phi$ if $|\phi|_{\mathfrak{M}} = 1$ and similarly for sets Γ.

We say that a Gödel-Mkrtychev model $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ *respects* a constant specification CS if

$$\mathcal{E}(c, \phi) = 1 \text{ for every } c : \phi \in CS.$$

Given a class of Gödel-Mkrtychev models C, we denote the subclass of all models respecting CS by C_{CS}.

Corresponding to the different additional justification principles, given by the axiom schemes (F), (!) and (?), we introduce respective model classes capturing them semantically.

[3]We define the empty conjunction to be $\top$.

Definition 3. A GM-model $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ is called a

(1) GMT-model if $\mathcal{E}(t,\phi) \leq |\phi|_{\mathfrak{M}}$ for all $t \in Jt$ and all $\phi \in \mathcal{L}_J$ (factivity),
(2) $\mathsf{GM4}$-model if $\mathcal{E}(t,\phi) \leq \mathcal{E}(!t, t : \phi)$ for all $t \in Jt$ and all $\phi \in \mathcal{L}_J$ (positive introspectivity),
(3) GMLP-model if (1) and (2) hold,
(4) $\mathsf{GM45}$-model if (2) holds and $\sim \mathcal{E}(t,\phi) \leq \mathcal{E}(?t, \neg t : \phi)$ for all $t \in Jt$ and all $\phi \in \mathcal{L}_J$ (negative introspectivity),
(5) $\mathsf{GMT45}$-model if (1) and (4) hold.

Based on Gödel-Mkrtychev models, there are now two different notions of semantic entailment, similarly as in propositional Gödel logic (see [31] for a survey).

Definition 4. For a class C of GM-models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, we say:

(1) Γ *1-entails* ϕ in C, written $\Gamma \models_{\mathsf{C}} \phi$, if for any $\mathfrak{M} \in \mathsf{C}$: $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \phi$;
(2) Γ *entails* ϕ in C, written $\Gamma \models_{\mathsf{C}\leq} \phi$, if for any $\mathfrak{M} \in \mathsf{C}$: $|\Gamma|_{\mathfrak{M}} \leq |\phi|_{\mathfrak{M}}$.

The main theorem about Gödel justification logics and the corresponding Gödel-Mkrtychev models is the Completeness Theorem with respect to both semantic entailment relations.

Theorem 3 (Completeness Theorem; $\mathcal{GJL}_{CS}$ and $\mathsf{GMJL_{CS}}$; [29])**.** *Let*

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$$

and CS be a constant specification for $\mathcal{GJL}_0$. Let

$$\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$$

be the class of Gödel-Mkrtychev models corresponding to $\mathcal{GJL}_0$. Then, for all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, the following are equivalent:

(1) $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$*;*
(2) $\Gamma \models_{\mathsf{GMJL_{CS}}\leq} \phi$*;*
(3) $\Gamma \models_{\mathsf{GMJL_{CS}}} \phi$*.*

We now turn to the alternative semantics given by Gödel-Fitting models, the $[0,1]$-valued analogues of the classical Fitting models. These classical models were introduced by Fitting in [12, 13] for the logic of proofs. The restrictions and extensions corresponding to other justification principles can be traced back to e.g. [3, 22] (although they may have appeared earlier).

Definition 5. A *Gödel-Fitting model* is a quadruple $\mathfrak{M} = \langle W, R, \mathcal{E}, e \rangle$ with

(1) $W \neq \emptyset$, the domain of $\mathfrak{M}$,
(2) $R : W \times W \to [0,1]$,
(3) $\mathcal{E} : W \times Jt \times \mathcal{L}_J \to [0,1]$,
(4) $e : W \times Var \to [0,1]$,

where $\mathcal{E}$ satisfies

(i) $\mathcal{E}(w, t, \phi \to \psi) \odot \mathcal{E}(w, s, \phi) \leq \mathcal{E}(w, t \cdot s, \psi)$,
(ii) $\mathcal{E}(w, t, \phi) \oplus \mathcal{E}(w, s, \phi) \leq \mathcal{E}(w, t + s, \phi)$,

for all $w \in W$, all $t, s \in Jt$ and all $\phi, \psi \in \mathcal{L}_J$.

We denote the class of all Gödel-Fitting models by GF and call a structure $\mathfrak{M}$ a GF-model if $\mathfrak{M} \in \mathsf{GF}$. An important notion for Gödel-Fitting models is being *accessibility-crisp*, that is Gödel-Fitting models where $R(w,v) \in \{0,1\}$ for all $w, v \in W$. For a class C of GF-models, we denote the subclass of all accessibility-crisp models in C by $\mathsf{C_c}$. Given a model $\mathfrak{M}$, we also denote its domain by $\mathcal{D}(\mathfrak{M})$.

For a GF-model $\mathfrak{M} = \langle W, R, \mathcal{E}, e \rangle$, we may define a local evaluation $|\cdot|^w_{\mathfrak{M}}$, relative to a world $w \in \mathcal{D}(\mathfrak{M})$, recursively on the structure of $\mathcal{L}_J$ as follows:

- $|\bot|^w_{\mathfrak{M}} := 0$; $|\top|^w_{\mathfrak{M}} := 1$;
- $|p|^w_{\mathfrak{M}} := e(w,p)$ for $p \in Var$;
- $|\phi \to \psi|^w_{\mathfrak{M}} := |\phi|^w_{\mathfrak{M}} \Rightarrow |\psi|^w_{\mathfrak{M}}$;
- $|\phi \wedge \psi|^w_{\mathfrak{M}} := |\phi|^w_{\mathfrak{M}} \odot |\psi|^w_{\mathfrak{M}}$;
- $|\phi \vee \psi|^w_{\mathfrak{M}} := |\phi|^w_{\mathfrak{M}} \oplus |\psi|^w_{\mathfrak{M}}$;
- $|t : \phi|^w_{\mathfrak{M}} := \mathcal{E}(w,t,\phi) \odot \inf_{v \in W}(R(w,v) \Rightarrow |\phi|^v_{\mathfrak{M}})$.

This extends to sets of formulae, locally, in a similar way as with Gödel-Mkrtychev models by $|\Gamma|^w_{\mathfrak{M}} := \inf_{\phi \in \Gamma} |\phi|^w_{\mathfrak{M}}$. We again write $(\mathfrak{M}, w) \models \phi$ if $|\phi|^w_{\mathfrak{M}} = 1$ and similarly for sets Γ.

Following [29], we may define a range of more restrictive classes of Gödel-Fitting models in analogy to the classical cases:

Definition 6. A Gödel-Fitting model $\mathfrak{M} = \langle W, R, \mathcal{E}, e\rangle$ is called a

(1) GFT-model if $R(w, w) = 1$ for all $w \in W$ (reflexivity),

(2) $\mathsf{GF4}$-model if
   (i) $\mathcal{E}(w, t, \phi) \odot R(w, v) \leq \mathcal{E}(v, t, \phi)$ (monotonicity),
   (ii) $R(w, v) \odot R(v, u) \leq R(w, u)$ (min-transitivity),
   (iii) $\mathcal{E}(w, t, \phi) \leq \mathcal{E}(w, !t, t : \phi)$ (positive introspectivity),
   for all $w, v, u \in W$, all $t \in Jt$ and all $\phi \in \mathcal{L}_J$,

(3) GFLP-model if it is a reflexive $\mathsf{GF4}$-model,

(4) $\mathsf{GF45}$-model if it is a $\mathsf{GF4}$-model satisfying
   (i) $\sim \mathcal{E}(w, t, \phi) \leq \mathcal{E}(w, ?t, \neg t : \phi)$ (negative introspectivity),
   (ii) $\mathcal{E}(w, t, \phi) \leq |t : \phi|^w_{\mathfrak{M}}$ (factivity),
   for all $w \in W$, all $t \in Jt$ and all $\phi \in \mathcal{L}_J$,

(5) $\mathsf{GFT45}$-model if it is a reflexive $\mathsf{GF45}$-model.

In similarity to Gödel-Mkrtychev models, given a constant specification CS, we say that a Gödel-Fitting model $\mathfrak{M}$ *respects* CS if

$$\mathcal{E}(w, c, \phi) = 1 \text{ for all } c : \phi \in CS \text{ and all } w \in W.$$

For a class C of Gödel-Fitting models, we denote the subclass of all models respecting a constant specification CS by C_{CS}.

Again in analogy to the case of Gödel-Mkrtychev models, one obtains two natural notions of semantical entailment for classes of Gödel-Fitting models.

Definition 7. Let C be a class of GF-models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. Then, we say:

(1) Γ *1-entails* ϕ in C, written $\Gamma \models_{\mathsf{C}} \phi$ if for any $\mathfrak{M} \in \mathsf{C}$ and any $w \in \mathcal{D}(\mathfrak{M})$: $(\mathfrak{M}, w) \models \Gamma$ implies $(\mathfrak{M}, w) \models \phi$;

(2) Γ *entails* ϕ in C, written $\Gamma \models_{\mathsf{C}\leq} \phi$ if for any $\mathfrak{M} \in \mathsf{C}$ and any $w \in \mathcal{D}(\mathfrak{M})$: $|\Gamma|^w_{\mathfrak{M}} \leq |\phi|^w_{\mathfrak{M}}$.

One now obtains a respective Completeness Theorem for the Gödel-Fitting semantics. A surprising addition, however, is that the models used in the Completeness Theorem can be restricted to be accessibility-crisp. This is similar to the Completeness Theorem of the $\Box$-fragment of the standard Gödel modal logics in [9] with respect to their possible world models (see section 2.2.2 for more detail).

Theorem 4 (Completeness Theorem; $\mathcal{GJL}_{CS}$ and $\mathsf{GFJL}_{\mathsf{CS}}$; [29]). *Let*

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\},$$

CS be a constant specification for $\mathcal{GJL}_0$ and let

$$\mathsf{GFJL} \in \{\mathsf{GF}, \mathsf{GFT}, \mathsf{GF4}, \mathsf{GFLP}, \mathsf{GF45}, \mathsf{GFT45}\}$$

be the class of Gödel-Fitting models corresponding to $\mathcal{GJL}_0$. For all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, the following are equivalent:

(1) $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$;
(2) $\Gamma \models_{\mathsf{GFJL}_{\mathsf{CS}}\leq} \phi$;
(3) $\Gamma \models_{\mathsf{GFJL}_{\mathsf{CS}}} \phi$;
(4) $\Gamma \models_{\mathsf{GFJL}_{\mathsf{CS}_c}} \phi$.

2.2. **Standard Gödel modal logics.** On the modal side, we fix a necessity-based modal language $\mathcal{L}_\Box$ by

$$\mathcal{L}_\Box : \phi ::= \bot \mid \top \mid p \mid (\phi \to \phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid \Box\phi.$$

We again write $\mathrm{sf}(\phi)$ for the set of all subformulae of ϕ (including ϕ).

In the later sections, in particular in the context of hypersequent calculi, we will need the notion of complexity $\mathrm{c}(\phi)$ of a modal formula ϕ. This is defined, recursively over $\mathcal{L}_\Box$, as follows:

- $\mathrm{c}(\bot) := \mathrm{c}(\top) := \mathrm{c}(p) := 0$;
- $\mathrm{c}(\phi \propto \psi) := 1 + \mathrm{c}(\phi) + \mathrm{c}(\psi)$ for $\propto \in \{\land, \lor, \to\}$;
- $\mathrm{c}(\Box\phi) := 1 + \mathrm{c}(\phi)$.

2.2.1. *Proof calculi.* The standard Gödel modal logics, as defined by Caicedo and Rodriguez in [9], have the following proof-theoretic descriptions via Hilbert-style calculi over $\mathcal{L}_\Box$.

Definition 8. $\mathcal{GK}_\Box$ is given by the following axiom schemes and rules:

(G): the axiom schemes of the calculus $\mathcal{G}$ in $\mathcal{L}_\Box$;
(K): $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
(Z): $\neg\neg\Box\phi \to \Box\neg\neg\phi$;
(MP): from $\phi \to \psi$ and ϕ, infer ψ;
$(N\Box)$: from $\vdash \phi$, infer $\vdash \Box\phi$.

We define the following axiomatic extensions of $\mathcal{GK}_\Box$:

(1) $\mathcal{GT}_\Box$ is the extension of $\mathcal{GK}_\Box$ by the axiom scheme $(T) : \Box\phi \to \phi$;
(2) $\mathcal{GK}4_\Box$ is the extension of $\mathcal{GK}_\Box$ by the axiom scheme $(4) : \Box\phi \to \Box\Box\phi$;
(3) $\mathcal{GS}4_\Box$ is the extension of $\mathcal{GK}_\Box$ by the axiom schemes (T) and (4).

Again, provability (under assumptions $\Gamma \subseteq \mathcal{L}_\Box$) of a formula $\phi \in \mathcal{L}_\Box$ in $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$ is defined as it is usually done in Hilbert-style calculi and is denoted by $\Gamma \vdash_{\mathcal{GML}_\Box} \phi$.

As before, the Deduction Theorem transfers from the propositional Gödel logic to the standard Gödel modal logics:

Lemma 5 (Deduction Theorem; [9]). *Let $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$ and $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_\Box$. Then*

$$\Gamma \cup \{\phi\} \vdash_{\mathcal{GML}_\Box} \psi \text{ iff } \Gamma \vdash_{\mathcal{GML}_\Box} \phi \to \psi.$$

The following lemma will later be generalized to fragments of the standard Gödel modal logics and is fundamental for a rule in the corresponding hypersequent calculi.

Lemma 6 ([9]). *Let $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$ and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$. Then*

$$\Gamma \vdash_{\mathcal{GML}_\Box} \phi \text{ implies } \Box\Gamma \vdash_{\mathcal{GML}_\Box} \Box\phi$$

where $\Box\Gamma := \{\Box\gamma \mid \gamma \in \Gamma\}$.

Indeed, using the axiom scheme (Z), we find another admissible rule of the standard Gödel modal logics which is also of importance in the corresponding formulations using hypersequent calculi, later on.

Lemma 7. *Let $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$ and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$. Then*

$$\neg\neg\Pi, \Gamma \vdash_{\mathcal{GML}_\Box} \phi \text{ implies } \neg\neg\Box\Pi, \Box\Gamma \vdash_{\mathcal{GML}_\Box} \Box\phi.$$

Proof. Lemma 6 applied to $\neg\neg\Pi, \Gamma \vdash_{\mathcal{GML}_\Box} \phi$ yields

$$\Box\neg\neg\Pi, \Box\Gamma \vdash_{\mathcal{GML}_\Box} \Box\phi$$

and using the axiom scheme (Z), we obtain

$$\neg\neg\Box\Pi, \Box\Gamma \vdash_{\mathcal{GML}_\Box} \Box\phi$$

from this. ☐

Remark 3. Note that

$$\neg\neg\Pi, \Gamma \vdash_{\mathcal{GML}_\Box} \phi \text{ iff } \vdash_{\mathcal{GML}_\Box} \left(\bigwedge_{\pi\in\Pi} \neg\neg\pi \to \bot\right) \vee \left(\bigwedge_{\gamma\in\Gamma} \gamma \to \phi\right).$$

The proof of this equivalence is similar to the proof given later for Lemma 55 regarding a similar statement for the Gödel justification logics.

2.2.2. *Semantics.* In [9], Caicedo and Rodriguez obtained a Completeness Theorem for these logics with respect to a natural semantics defined over model classes of $[0,1]$-valued generalization of the classical modal Kripke models, called Gödel-Kripke models:

Definition 9. A *Gödel-Kripke model* is a triple $\mathfrak{M} = \langle W, R, e\rangle$ where

(1) $W \neq \emptyset$,
(2) $R : W \times W \to [0,1]$,
(3) $e : W \times Var \to [0,1]$.

We denote the class of all Gödel-Kripke models by GK and say that $\mathfrak{M}$ is a GK-model if $\mathfrak{M} \in \mathsf{GK}$. GK-models capture the base logic $\mathcal{GK}_\Box$. Again, given a class C of GK-models, we denote its subclass of all accessibility-crisp models (defined as with the Gödel-Fitting models) by $\mathsf{C_c}$.

In similarity to Gödel-Fitting models, there is a natural extension of the function e to the set of formulae $\mathcal{L}_\Box$ via the function $|\cdot|^w_{\mathfrak{M}} : \mathcal{L}_\Box \to [0,1]$, parameterized by worlds $w \in W$, which is (recursively) defined as follows:

- $|\bot|^w_{\mathfrak{M}} := 0$; $|\top|^w_{\mathfrak{M}} := 1$;
- $|p|^w_{\mathfrak{M}} := e(w,p)$ for $p \in Var$;
- $|\phi \to \psi|^w_{\mathfrak{M}} := |\phi|^w_{\mathfrak{M}} \Rightarrow |\psi|^w_{\mathfrak{M}}$;
- $|\phi \wedge \psi|^w_{\mathfrak{M}} := |\phi|^w_{\mathfrak{M}} \odot |\psi|^w_{\mathfrak{M}}$;
- $|\phi \vee \psi|^w_{\mathfrak{M}} := |\phi|^w_{\mathfrak{M}} \oplus |\psi|^w_{\mathfrak{M}}$;
- $|\Box\phi|^w_{\mathfrak{M}} := \inf_{v\in W}(R(w,v) \Rightarrow |\phi|^v_{\mathfrak{M}})$.

By $|\Gamma|^w_{\mathfrak{M}} := \inf_{\gamma\in\Gamma} |\gamma|^w_{\mathfrak{M}}$, we again extend the evaluation function to sets of formulae $\Gamma \subseteq \mathcal{L}_\Box$ and write $(\mathfrak{M}, w) \models \phi$ for $|\phi|^w_{\mathfrak{M}} = 1$ and similarly for sets.

Following [9], we may define natural refinements of the class of Gödel-Kripke models to capture the other proof-calculi extending $\mathcal{GK}_\Box$.

Definition 10. Let $\mathfrak{M} = \langle W, R, e\rangle$ be a GK-model. We say that $\mathfrak{M}$ is a

(1) GT-model if $R(w, w) = 1$ for all $w \in W$ (reflexivity),
(2) GK4-model if $R(w, v) \odot R(v, u) \leq R(w, u)$ for all $w, v, u \in W$ (min-transitivity),
(3) GS4-model if its both a GT- and GK4-model.

As before with the Gödel justification logics, we have two natural forms of semantic consequence over model classes.

Definition 11. Let C be a class of GK-models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$. We write

(1) $\Gamma \models_{\mathsf{C}} \phi$ if $\forall \mathfrak{M} \in \mathsf{C} : \forall w \in W : (\mathfrak{M}, w) \models \Gamma$ implies $(\mathfrak{M}, w) \models \phi$,
(2) $\Gamma \models_{\mathsf{C}\leq} \phi$ if $\forall \mathfrak{M} \in \mathsf{C} : \forall w \in W : |\Gamma|^w_{\mathfrak{M}} \leq |\phi|^w_{\mathfrak{M}}$.

Caicedo and Rodriguez then obtained the following Completeness Theorem for the various model classes and proof systems where, as commented on in the context of the Completeness Theorem for Gödel-Fitting models, accessibility-crisp models suffice.

Theorem 8 (Completeness Theorem; $\mathcal{GML}_\Box$ and GML; [9])**.** *Let*

$$\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$$

and let

$$\mathsf{GML} \in \{\mathsf{GK}, \mathsf{GT}, \mathsf{GK4}, \mathsf{GS4}\}$$

be the corresponding class of GK*-models. Then, for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$, the following are equivalent:*

(1) $\Gamma \vdash_{\mathcal{GML}_\Box} \phi$*;*
(2) $\Gamma \models_{\mathsf{GML}\leq} \phi$*;*
(3) $\Gamma \models_{\mathsf{GML}} \phi$*;*
(4) $\Gamma \models_{\mathsf{GML}_c} \phi$*.*

2.3. **Forgetful projection.** A natural projection from the explicit modal language $\mathcal{L}_J$ to the unexplicit language $\mathcal{L}_\Box$ is the function which replaces every explicit modality "t :" by $\Box$. We define this so-called *forgetful projection* $\nu : \mathcal{L}_J \to \mathcal{L}_\Box$ formally by recursion on the structure of $\mathcal{L}_J$ as follows:

- $p \mapsto p$ for $p \in Var$;
- $\bot \mapsto \bot$; $\top \mapsto \top$;
- $\phi \propto \psi \mapsto \phi^\nu \propto \psi^\nu$ for $\propto \in \{\wedge, \vee, \to\}$;
- $t : \phi \mapsto \Box\phi^\nu$.

We may extend ν to sets of formulae $\Gamma \subseteq \mathcal{L}_J$ via $\Gamma^\nu := \{\phi^\nu \mid \phi \in \Gamma\}$.

We use ν in difference to the commonly used $\circ$ to denote the forgetful projection as we use $\circ$ for function composition later on.

Remark 4. For the various axioms of Gödel justification logics, we obtain the following forgetful projections:

(1) $(t : (\phi \to \psi) \to (s : \phi \to [t \cdot s] : \psi))^\nu = \Box(\phi^\nu \to \psi^\nu) \to (\Box\phi^\nu \to \Box\psi^\nu)$;
(2) $(t : \phi \to [t + s] : \phi)^\nu = \Box\phi^\nu \to \Box\phi^\nu$; $(s : \phi \to [t + s] : \phi)^\nu = \Box\phi^\nu \to \Box\phi^\nu$;
(3) $(t : \phi \to \phi)^\nu = \Box\phi^\nu \to \phi^\nu$;
(4) $(t : \phi \to !t : t : \phi)^\nu = \Box\phi^\nu \to \Box\Box\phi^\nu$.

Note that the cases in (2) are instances of a propositional tautology, while (1), (3) and (4) are instances of the various axioms of the standard Gödel modal logics, all in the language of $\mathcal{L}_\Box$. This results in the following theorem.

Theorem 9. *Let $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0\}$, CS be a constant specification for $\mathcal{GJL}_0$ and $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$ be the corresponding standard Gödel modal logic. Then, for all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$: $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$ implies $\Gamma^\nu \vdash_{\mathcal{GML}_\Box} \phi^\nu$.*

The proof of the theorem is a straightforward induction on the length of the proof.

3. Realization fails without factivity

In the following, let TCS be the total constant specification for $\mathcal{GJ}45_0$ and $x \in [0,1]$. We define the structure $\mathfrak{M}_x := \langle \mathcal{E}_x, e_x \rangle$ by $e_x(p) := x$ for any $p \in Var$ and

$$\mathcal{E}_x(t,\phi) := \begin{cases} 1 & \text{if } \vdash_{\mathcal{GJ}45_{TCS}} \phi \text{ and } \vdash_{\mathcal{GJ}45_{TCS}} t:\phi, \\ x & \text{else,} \end{cases}$$

for any $t \in Jt$ and any $\phi \in \mathcal{L}_J$. It is easy to see that, given $\phi \in \mathcal{L}_J$, we have $|\phi|_{\mathfrak{M}_x} \in \{0, x, 1\}$.

Now, $\mathfrak{M}_x$ is indeed a well-defined Gödel-Mkrtychev model (for certain x):

Lemma 10. *If $x \in (0,1]$, then $\mathfrak{M}_x$ is a $\mathsf{GM45_{TCS}}$-model.*

Proof. We verify the conditions for "being a $\mathsf{GM45_{TCS}}$-model" from Definition 2 and 3 for $\mathfrak{M}_x$:

(a) We show that $\mathfrak{M}_x$ respects TCS. We have $\vdash_{\mathcal{GJ}45_{TCS}} c:\phi$ whenever we have $c:\phi \in TCS$. By definition, ϕ is either an axiom instance or $\phi = c':\psi$ for some $c' \in C$ and some formula ψ with $\phi \in TCS$ by downward closure. Either way $\vdash_{\mathcal{GJ}45_{TCS}} \phi$ and thus we have $\mathcal{E}_x(c,\phi) = 1$ for any such $c:\phi$.

(b) We show that $\mathfrak{M}_x$ satisfies condition (ii) from Definition 2. Let $\phi \in \mathcal{L}_J$ and $t, s \in Jt$. If $\mathcal{E}_x(t,\phi) \oplus \mathcal{E}_x(s,\phi) = x$, then (ii) is immediately satisfied. Thus suppose $\mathcal{E}_x(t,\phi) \oplus \mathcal{E}_x(s,\phi) = 1$, i.e. per definition $\mathcal{E}_x(t,\phi) = 1$ or $\mathcal{E}_x(s,\phi) = 1$. In either case $\vdash_{\mathcal{GJ}45_{TCS}} \phi$ and additionally $\vdash_{\mathcal{GJ}45_{TCS}} t:\phi$ or $\vdash_{\mathcal{GJ}45_{TCS}} s:\phi$. Either way, by the axiom scheme $(+)$ and the rule (MP), we have $\vdash_{\mathcal{GJ}45_{TCS}} [t+s]:\phi$, and therefore $\mathcal{E}_x(t+s,\phi) = 1$.

(c) We show that $\mathfrak{M}_x$ satisfies condition (i) from Definition 2. Fix $\phi, \psi \in \mathcal{L}_J$ and $t, s \in Jt$. If $\mathcal{E}_x(t, \phi \to \psi) \odot \mathcal{E}_x(s,\phi) = x$, then the condition is immediately satisfied. Thus, suppose $\mathcal{E}_x(t, \phi \to \psi) \odot \mathcal{E}_x(s,\phi) = 1$, i.e. $\mathcal{E}_x(t, \phi \to \psi) = \mathcal{E}_x(s,\phi) = 1$ and therefore $\vdash_{\mathcal{GJ}45_{TCS}} \phi \to \psi$ and $\vdash_{\mathcal{GJ}45_{TCS}} t:(\phi \to \psi)$ as well as $\vdash_{\mathcal{GJ}45_{TCS}} \phi$ and $\vdash_{\mathcal{GJ}45_{TCS}} s:\phi$. By (MP) and the axiom scheme (J), we have $\vdash_{\mathcal{GJ}45_{TCS}} \psi$ as well as $\vdash_{\mathcal{GJ}45_{TCS}} [t \cdot s]:\psi$, i.e. $\mathcal{E}_x(t \cdot s, \psi) = 1$.

(d) We show that $\mathfrak{M}_x$ satisfies condition (4) from Definition 3. For this, we first show condition (2) from Definition 3. Let $\phi \in \mathcal{L}_J$ and $t \in Jt$ be arbitrary. If $\mathcal{E}_x(t,\phi) = x$, then we immediately obtain $\mathcal{E}_x(t,\phi) \leq \mathcal{E}_x(!t, t:\phi)$. Thus, suppose $\mathcal{E}_x(t,\phi) = 1$, then $\vdash_{\mathcal{GJ}45_{TCS}} \phi$ and $\vdash_{\mathcal{GJ}45_{TCS}} t:\phi$. The latter implies $\vdash_{\mathcal{GJ}45_{TCS}} !t:t:\phi$ by the axiom scheme (!) and (MP), which yields $\mathcal{E}_x(!t, t:\phi) = 1$. Thus, $\mathfrak{M}_x$ satisfies condition (2) from Definition 3.

For the latter part of condition (4), note that we always have $\mathcal{E}_x(t,\phi) \in \{x, 1\}$, i.e. as $x > 0$ we have $\sim \mathcal{E}_x(t,\phi) = 0$ and thus, for any $\phi \in \mathcal{L}_J$ and any $t \in Jt$, we have $\sim \mathcal{E}_x(t,\phi) \leq \mathcal{E}_x(?t, \neg t:\phi)$. Hence $\mathfrak{M}_x$ satisfies condition (4) from Definition 3.

□

$\mathfrak{M}_x$ now serves as a counter model for realization instances of the modal axiom (Z).

Lemma 11. *For any $\phi \in \mathcal{L}_J$ such that $\nvdash_{\mathcal{GJ}45_{TCS}} \neg\neg\phi$ and any $t, s \in Jt$:*

$$\nvdash_{\mathcal{GJ}45_{TCS}} \neg\neg t:\phi \to s:\neg\neg\phi.$$

Proof. Suppose $\nvdash_{\mathcal{GJ}45_{TCS}} \neg\neg\phi$ for $\phi \in \mathcal{L}_J$ and let $t, s \in Jt$ as well as $x \in (0,1)$. As $\mathcal{E}_x(t,\phi) > 0$, we have $|\neg\neg t:\phi|_{\mathfrak{M}_x} = 1$ by the semantical evaluation of $\neg$ by $\sim$. However, we also have

$$|s:\neg\neg\phi|_{\mathfrak{M}_x} = \mathcal{E}_x(s, \neg\neg\phi) = x < 1$$

as $\nvdash_{\mathcal{GJ}45_{TCS}} \neg\neg\phi$. Thus, we get

$$|\neg\neg t:\phi \to s:\neg\neg\phi|_{\mathfrak{M}_x} = x < 1.$$

By Lemma 10, $\mathfrak{M}_x$ is a $\mathsf{GM45_{TCS}}$-model. Per definition, this leads to

$$\not\models_{\mathsf{GM45}_{TCS}} \neg\neg t:\phi \to s:\neg\neg\phi,$$

from which we obtain

$$\nvdash_{\mathcal{GJ}45_{TCS}} \neg\neg t:\phi \to s:\neg\neg\phi$$

by Theorem 3. □

By this lemma, there is no valid (realized) formula structured like the (Z)-axiom where the *instantiating formula* is such that its double-negation projection is not provable (or valid). As, e.g., the double negation of any propositional variable p is never provable, there is no realization of $\neg\neg\Box p \to \Box\neg\neg p$. This results in the following two theorems. Here, and in the following, we write $Th_{\mathcal{S}} := \{\phi \in \mathcal{L} \mid \vdash_{\mathcal{S}} \phi\}$ for the set of theorems of a given proof system $\mathcal{S}$ over a language $\mathcal{L}$.

Theorem 12. *For any constant specification CS for $\mathcal{GJ}_0$: $(Th_{\mathcal{GJ}_{CS}})^{\nu} \subsetneq Th_{\mathcal{GK}_{\Box}}$.*

Theorem 13. *For any constant specification CS for $\mathcal{GJ}4_0$: $(Th_{\mathcal{GJ}4_{CS}})^\nu \subsetneq Th_{\mathcal{GK}4_\Box}$.*

In fact, $\mathcal{GJ}4_{CS}$ does not even realize $\mathcal{GK}_\Box$, as the problem with the axiom scheme (Z) remains.

It is also important to note that, in the proof of Lemma 11, it is crucial that Gödel-Mkrtychev models are many-valued as $x \in (0,1)$ is necessary. Making $\mathfrak{M}_x$ crisp by moving x to 1 makes any instance of $\neg\neg t : \phi \to s : \neg\neg\phi$ valid in $\mathfrak{M}_x$. With moving x to 0, we obtain that $\mathfrak{M}_0$ is only a $\mathsf{GM4}$-model. But even in this case, at least some instance of $\neg\neg t : \phi \to s : \neg\neg\phi$ is valid in $\mathfrak{M}_0$ (for any ϕ): crisp Gödel-Mkrtychev models correspond to classical Mkrtychev models, and in the classical modal logics we have that

$$\neg\neg\Box\phi \to \Box\neg\neg\phi$$

is, of course, valid. Thus, $\neg\neg\Box\phi \to \Box\neg\neg\phi$ has a classical realization and this realization is valid in all classical Mkrtychev models and hence valid in all crisp Gödel-Mkrtychev models, in particular in $\mathfrak{M}_0$.

In Lemma 11, the condition $\nvdash_{\mathcal{GJ}45_{TCS}} \neg\neg\phi$ is necessary, as if $\vdash_{\mathcal{GJ}45_{TCS}} \neg\neg\phi$, then by Internalization (Corollary 1, as TCS is axiomatically appropriate), we have $\vdash_{\mathcal{GJ}45_{TCS}} s : \neg\neg\phi$ for some $s \in Jt$ and then by propositional reasoning in $\mathcal{GJ}45_{TCS}$, we would obtain

$$\vdash_{\mathcal{GJ}45_{TCS}} \neg\neg t : \phi \to s : \neg\neg\phi$$

for any t.

4. Realization fails with factivity

Using the same model construction, we can also show that $\mathcal{GJT}_{CS}$ and $\mathcal{GLP}_{CS}$ do not realize $\mathcal{GT}_\Box$ and $\mathcal{GS}4_\Box$. However, we need another Completeness Theorem for this. This is because the factivity condition $\mathcal{E}(t,\phi) \leq |\phi|_{\mathfrak{M}}$ from Definition 3 fails for $\mathfrak{M}_x$: per definition, $\mathcal{E}_x(t,\phi) > 0$ for any $t \in Jt$ and any $\phi \in \mathcal{L}_J$, hence also $\mathcal{E}_x(t,\bot) > 0 = |\bot|_{\mathfrak{M}_x}$.

We thus resort to an alternative definition of semantical evaluations (and, induced from this, to an alternative definition of (semantic) consequence) in Gödel-Mkrtychev models. Mkrtychev, in his paper [28], called the corresponding classical concept *pre-models* and our situation is quite similar to the one in Kuznets' works [21, 22] where he also resorts to pre-models to provide counter-model constructions in the presence of the axiom scheme (F) (in investigations into computational complexity, however).

4.1. **An alternative Completeness Theorem.** For a Gödel-Mkrtychev model $\mathfrak{M} = \langle \mathcal{E}, e\rangle$, we define the alternative evaluation function $|\cdot|^*_{\mathfrak{M}}$ as follows:

- $|\bot|^*_{\mathfrak{M}} := 0$; $|\top|^*_{\mathfrak{M}} := 1$;
- $|p|^*_{\mathfrak{M}} := e(p)$ for $p \in Var$;
- $|\phi \to \psi|^*_{\mathfrak{M}} := |\phi|^*_{\mathfrak{M}} \Rightarrow |\psi|^*_{\mathfrak{M}}$;
- $|\phi \wedge \psi|^*_{\mathfrak{M}} := |\phi|^*_{\mathfrak{M}} \odot |\psi|^*_{\mathfrak{M}}$;
- $|\phi \vee \psi|^*_{\mathfrak{M}} := |\phi|^*_{\mathfrak{M}} \oplus |\psi|^*_{\mathfrak{M}}$;
- $|t : \phi|^*_{\mathfrak{M}} := \mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{M}}$.

We extend this evaluation to sets of formulae Γ in the same way as before by setting $|\Gamma|^*_{\mathfrak{M}} = \inf_{\phi\in\Gamma} |\phi|^*_{\mathfrak{M}}$. Again, we write $\mathfrak{M} \models^* \phi$ if $|\phi|^*_{\mathfrak{M}} = 1$ and similarly for sets Γ. The corresponding definition of semantical (1-)entailment then follows naturally:

Definition 12. Let C be a class of GM-models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We write $\Gamma \models^*_{\mathsf{C}} \phi$ if for any $\mathfrak{M} \in \mathsf{C}$: $\mathfrak{M} \models^* \Gamma$ implies $\mathfrak{M} \models^* \phi$.

The following two lemmas now establish the equivalence between $\models^*_{\mathsf{C}'}$ and $\models_{\mathsf{C}}$ for C being one of the classes GMT or GMLP and C' being the class C without requiring the factivity condition (i.e. GM or $\mathsf{GM4}$, respectively). The lemmas and proofs are fuzzy replicas of analogous classical results found in [21, 28].

Lemma 14. *For every $\mathfrak{M} \in \mathsf{GMT}$ (or $\mathfrak{M} \in \mathsf{GMLP}$), there is an $\mathfrak{N} \in \mathsf{GM}$ (or $\mathfrak{N} \in \mathsf{GM4}$, respectively) such that $|\phi|_{\mathfrak{M}} = |\phi|^*_{\mathfrak{N}}$ for every $\phi \in \mathcal{L}_J$.*

Proof. Let $\mathfrak{M} = \langle \mathcal{E}, e\rangle \in \mathsf{GMT}$ (or GMLP) and set $\mathfrak{N} := \mathfrak{M}$. Then naturally $\mathfrak{N} \in \mathsf{GM}$ (or $\mathsf{GM4}$). We show the claim by induction on $\mathcal{L}_J$. The propositional cases are clear, so let $\phi \in \mathcal{L}_J$ be such that $|\phi|_{\mathfrak{M}} = |\phi|^*_{\mathfrak{N}}$ and let $t \in Jt$ be arbitrary. We have

$$\begin{aligned} |t : \phi|^*_{\mathfrak{N}} &= \mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}} \\ &= \mathcal{E}(t,\phi) \odot |\phi|_{\mathfrak{M}} \\ &= \mathcal{E}(t,\phi) \\ &= |t : \phi|_{\mathfrak{M}} \end{aligned}$$

where the third equality follows from the definition of GMT (or GMLP) which requires $\mathcal{E}(t,\phi) \leq |\phi|_{\mathfrak{M}}$. ☐

Lemma 15. *For every $\mathfrak{N} \in \mathsf{GM}$ (or $\mathfrak{N} \in \mathsf{GM4}$), there is an $\mathfrak{M} \in \mathsf{GMT}$ (or $\mathfrak{M} \in \mathsf{GMLP}$, respectively) such that $|\phi|^*_{\mathfrak{N}} = |\phi|_{\mathfrak{M}}$ for every $\phi \in \mathcal{L}_J$.*

Proof. Suppose $\mathfrak{N} = \langle \mathcal{E}, e\rangle \in \mathsf{GM}$ (or $\mathsf{GM4}$) and define $\mathfrak{M} := \langle \mathcal{E}', e\rangle$ through $\mathcal{E}'(t,\phi) := \mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}}$. We first show $|\phi|^*_{\mathfrak{N}} = |\phi|_{\mathfrak{M}}$ for every $\phi \in \mathcal{L}_J$ by induction on $\mathcal{L}_J$. Again, the propositional cases are clear. So let ϕ satisfy the claim and let $t \in Jt$ be arbitrary. By definition, we have

$$|t : \phi|_{\mathfrak{M}} = \mathcal{E}'(t,\phi) = \mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}} = |t : \phi|^*_{\mathfrak{N}}.$$

It remains to show that $\mathfrak{M} \in \mathsf{GMT}$ (or GMLP). For this, we first note that

$$\begin{aligned}
\mathcal{E}'(t,\phi \to \psi) \odot \mathcal{E}'(s,\phi) &= (\mathcal{E}(t,\phi \to \psi) \odot |\phi \to \psi|^*_{\mathfrak{N}}) \odot (\mathcal{E}(s,\phi) \odot |\phi|^*_{\mathfrak{N}}) \\
&= (\mathcal{E}(t,\phi \to \psi) \odot \mathcal{E}(s,\phi)) \odot (|\phi \to \psi|^*_{\mathfrak{N}} \odot |\phi|^*_{\mathfrak{N}}) \\
&\leq \mathcal{E}(t \cdot s,\psi) \odot |\psi|^*_{\mathfrak{N}} \\
&= \mathcal{E}'(t \cdot s,\psi)
\end{aligned}$$

as well as

$$\begin{aligned}
\mathcal{E}'(t,\phi) \oplus \mathcal{E}'(s,\phi) &= (\mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}}) \oplus (\mathcal{E}(s,\phi) \odot |\phi|^*_{\mathfrak{N}}) \\
&= (\mathcal{E}(t,\phi) \oplus \mathcal{E}(s,\phi)) \odot |\phi|^*_{\mathfrak{N}} \\
&\leq \mathcal{E}(t + s,\phi) \odot |\phi|^*_{\mathfrak{N}} \\
&= \mathcal{E}'(t + s,\phi).
\end{aligned}$$

For the factivity condition, we naturally have

$$\mathcal{E}'(t,\phi) = \mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}} \leq |\phi|^*_{\mathfrak{N}} = |\phi|_{\mathfrak{M}}$$

where the last equality follows from the before proved adequacy of $\mathfrak{M}$ for $\mathfrak{N}$. If $\mathfrak{N}$ is a $\mathsf{GM4}$-model, then also $\mathcal{E}(t,\phi) \leq \mathcal{E}(!t, t : \phi)$ and therefore

$$\begin{aligned}
\mathcal{E}'(t,\phi) &= \mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}} \\
&\leq \mathcal{E}(!t, t : \phi) \odot |t : \phi|^*_{\mathfrak{N}} \\
&= \mathcal{E}'(!t, t : \phi)
\end{aligned}$$

where the inequality follows from the fact that $\mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}} \leq \mathcal{E}(!t, t : \phi)$ as well as $\mathcal{E}(t,\phi) \odot |\phi|^*_{\mathfrak{N}} = |t : \phi|^*_{\mathfrak{N}}$. ☐

Note that the previous definition regarding whether a model $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ respects a constant specification CS (i.e., that $\mathcal{E}(c,\phi) = 1$ for $c : \phi \in CS$) is still feasible as a definition for respecting a constant specification in this context of the new consequence relation $\models^*$. With feasible, we mean that the defining equivalence

$$\mathfrak{M} \text{ respects } CS \text{ iff } \mathfrak{M} \models CS$$

extends to the satisfaction relation $\models^*$ under certain conditions: let $\mathcal{GJL}_0 \in \{\mathcal{GJT}_0, \mathcal{GLP}_0\}$ and $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GM4}\}$ be the corresponding class of non-factive GM-models, respectively. Also, let CS be a constant specification for $\mathcal{GJL}_0$. Then, for a GMJL-model $\mathfrak{M}$, it holds that $\mathfrak{M}$ respects CS iff $\mathfrak{M} \models^* CS$.

The direction from right to left is immediate and for the direction from left to right, note that by definition every formula in CS is of the form $c_{i_n} : \cdots : c_{i_1} : \phi$ where ϕ is an axiom instance of $\mathcal{GJL}_0$. By Lemma 15, ϕ is valid in $\mathfrak{M}$ with respect to $\models^*$. Thus, by $\mathcal{E}(c_{i_1},\phi) = 1$ (as $\mathfrak{M}$ respects CS), we have $\mathfrak{M} \models^* c_{i_1} : \phi$. Iterating this argument gives

$$\mathfrak{M} \models^* c_{i_n} : \cdots : c_{i_1} : \phi.$$

We therefore have the following additional information on the two previous lemmas:

- in Lemma 14, if $\mathfrak{M}$ respects CS, then $\mathfrak{N}$ respects CS;
- in Lemma 15, if $\mathfrak{N}$ respects CS, then $\mathfrak{M}$ respects CS.

Using these observations and the above lemmas, we can derive the following Completeness Theorem.

Theorem 16. *Let $\mathcal{GJL}_0 \in \{\mathcal{GJT}_0, \mathcal{GLP}_0\}$ and $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GM4}\}$ be the respective class of non-factive GM-models. For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$: $\Gamma \models^*_{\mathsf{GMJL_{CS}}} \phi$ iff $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$.*

Proof. By the standard Completeness Theorem, Theorem 3, it suffices to show the equivalence of $\models_{\mathsf{GMJLT_{CS}}}$ and $\models^*_{\mathsf{GMJL_{CS}}}$ where $\mathsf{GMJLT} \in \{\mathsf{GMT}, \mathsf{GMLP}\}$ is the class of factive GM-models corresponding to $\mathcal{GJL}_0$.

Suppose $\Gamma \models^*_{\mathsf{GMJL_{CS}}} \phi$, i.e. for every $\mathfrak{M} \in \mathsf{GMJL_{CS}}$, if $\mathfrak{M} \models^* \Gamma$, then $\mathfrak{M} \models^* \phi$. By Lemma 14, for every $\mathfrak{N} \in \mathsf{GMJLT_{CS}}$, we have $\mathfrak{N} \models \Gamma$, then $\mathfrak{N} \models \phi$. Thus, we get $\Gamma \models_{\mathsf{GMJLT_{CS}}} \phi$.

For the reverse, suppose $\Gamma \models_{\mathsf{GMJLT_{CS}}} \phi$, i.e. for every $\mathfrak{M} \in \mathsf{GMJLT_{CS}}$, if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \phi$. Again, now by Lemma 15, for every $\mathfrak{N} \in \mathsf{GMJL_{CS}}$, we have $\mathfrak{N} \models^* \Gamma$, then $\mathfrak{N} \models^* \phi$. Hence, we have $\Gamma \models^*_{\mathsf{GMJL_{CS}}} \phi$.

Note, for both directions, the above discussion on whether the models respect the constant specification CS. □

4.2. **$\mathcal{GJT}_{CS}$ and $\mathcal{GLP}_{CS}$ do not realize $\mathcal{GT}_\Box$ and $\mathcal{GS}4_\Box$.** Here, let TCS be the total constant specification for $\mathcal{GLP}_0$. Again, with $x \in [0,1]$, we define the model $\mathfrak{M}'_x := \langle \mathcal{E}'_x, e_x \rangle$ with e_x as before and

$$\mathcal{E}'_x(t,\phi) := \begin{cases} 1 & \text{if } \vdash_{\mathcal{GLP}_{TCS}} \phi \text{ and } \vdash_{\mathcal{GLP}_{TCS}} t:\phi, \\ x & \text{else,} \end{cases}$$

for any $t \in Jt$ and any $\phi \in \mathcal{L}_J$. Again, we get the following lemma, however now for TCS being the total constant specification for $\mathcal{GLP}_0$.

Lemma 17. *If $x \in (0,1)$, then $\mathfrak{M}'_x$ is a $\mathsf{GM4_{TCS}}$-model.*

The proof is almost identical to the one of Lemma 10 and thus omitted.

With a slightly changed proof, modified for the alternative semantics, we also obtain the next lemma in analogy to Lemma 11. Here, however, we have to restrict ourselves to propositional variables in $\mathcal{L}_J$, as it is relatively hard to control the truth value of compound statements containing justifications in the new semantics.

Lemma 18. *For any $p \in Var$ and any $t, s \in Jt$:*

$$\not\vdash_{\mathcal{GLP}_{TCS}} \neg\neg t:p \rightarrow s:\neg\neg p.$$

Proof. Let $p \in Var$ and $t, s \in Jt$ as well as $x \in (0,1)$. Then, naturally $\not\vdash_{\mathcal{GLP}_{TCS}} p$ and $\not\vdash_{\mathcal{GLP}_{TCS}} \neg\neg p$. Thus, $|t:p|^*_{\mathfrak{M}'_x} = \mathcal{E}'_x(t,p) \odot |p|^*_{\mathfrak{M}'_x} = \mathcal{E}'_x(t,p) \odot e_x(p) = x \odot x = x \in (0,1)$.

As $x > 0$, we have $|\neg\neg t:p|^*_{\mathfrak{M}'_x} = 1$ as before. However, we get

$$|s:\neg\neg p|^*_{\mathfrak{M}'_x} = \mathcal{E}'_x(s,\neg\neg p) \odot |\neg\neg p|^*_{\mathfrak{M}'_x} = \mathcal{E}'_x(s,\neg\neg p) \odot \sim^2 e_x(p) = x \odot 1 = x < 1$$

as $\not\vdash_{\mathcal{GLP}_{TCS}} \neg\neg p$ and $e_x(p) = x > 0$, i.e. $\sim^2 e_x(p) = 1$. Thus, we obtain

$$|\neg\neg t:p \rightarrow s:\neg\neg p|^*_{\mathfrak{M}'_x} = x < 1$$

By Lemma 17, $\mathfrak{M}'_x$ is a $\mathsf{GM4_{TCS}}$-model and thus, we have

$$\not\models^*_{\mathsf{GM4_{TCS}}} \neg\neg t:p \rightarrow s:\neg\neg p$$

by definition. Thus, Theorem 16 implies

$$\not\vdash_{\mathcal{GLP}_{TCS}} \neg\neg t:p \rightarrow s:\neg\neg p.$$

□

As before, we obtain the following two theorems.

Theorem 19. *For any constant specification CS of $\mathcal{GJT}_0$: $(Th_{\mathcal{GJT}_{CS}})^\nu \subsetneq Th_{\mathcal{GT}_\Box}$.*

Theorem 20. *For any constant specification CS of $\mathcal{GLP}_0$: $(Th_{\mathcal{GLP}_{CS}})^\nu \subsetneq Th_{\mathcal{GS}4_\Box}$.*

As with $\mathcal{GJ}4_0$, also $\mathcal{GJT}_{CS}$ and $\mathcal{GLP}_{CS}$ do not even realize $\mathcal{GK}_\Box$.

The main focus of the rest of the paper will be to modify the right- and left-hand sides of the strict inclusions in the Theorems 12, 13, 19 and 20 to induce equalities: on the one hand, we study the fragments of the various standard Gödel modal logics which *are* realized by the standard Gödel justification logics; on the other hand, we study extensions of the standard Gödel justification logics which are strong enough to realize the usual standard Gödel modal logics.

5. Positive justification logics

To model explicit epistemic inference in the style of the modal axiom scheme (Z), we introduce a new operator ϑ into the language of justification terms. That is, we define the augmented set of justification terms

$$Jt_\vartheta : t ::= x \mid c \mid [t \cdot t] \mid [t+t] \mid !t \mid ?t \mid \vartheta t$$

and, correspondingly, the following extended language of propositional justification logics *with* ϑ, that is

$$\mathcal{L}_{\vartheta J} : \phi ::= \bot \mid \top \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid t:\phi$$

where now $t \in Jt_\vartheta$ and $p \in Var$ as before. The functions var, jvar and sf of course naturally extend to these languages.

Now, to proof-theoretically model explicit inference in the style of (Z), we extend the previous proof systems by a new axiom scheme defining the operator ϑ. For this, let $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$. We define $\mathcal{PGJL}_0$ as the expansion of $\mathcal{GJL}_0$ (in the new language $\mathcal{L}_{\vartheta J}$) by the axiom scheme

$$(P) : \neg\neg t:\phi \rightarrow \vartheta t:\neg\neg\phi.$$

The concept of a constant specification naturally generalizes to these logics over the language $\mathcal{L}_{\vartheta J}$ and followingly, for a constant specification CS for $\mathcal{PGJL}_0$, we write $\mathcal{PGJL}_{CS}$ for the extension of $\mathcal{PGJL}_0$ with the corresponding rule (CS) as before.

Remark 5. Naturally, these positive Gödel justification logics also enjoy corresponding analogues of the classical Deduction Theorem as well as of the Lifting Lemma (and, consequently, of the Internalization Property) for axiomatically appropriate constant specifications.

It is clear that the forgetful projection (lifted to $\mathcal{L}_{\vartheta J}$ and which we still denote by ν) of an instance of the axiom scheme (P) is an instance of the axiom scheme (Z). We thus have the following theorem by a straightforward induction on the length of the proof as a generalization of Lemma 33.

Lemma 21. *Let* $\mathcal{PGJL}_0 \in \{\mathcal{PGJ}_0, \mathcal{PGJT}_0, \mathcal{PGJ}4_0, \mathcal{PGLP}_0\}$, CS *be a constant specification for* $\mathcal{PGJL}_0$ *and* $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$ *be the corresponding Gödel modal logic. Then, for all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\vartheta J}$*:* $\Gamma \vdash_{\mathcal{PGJL}_{CS}} \phi$ *implies* $\Gamma^\nu \vdash_{\mathcal{GML}_\Box} \phi^\nu$*.*

5.1. **Semantics.** Next, we introduce specific kinds of Gödel-Mkrtychev and Gödel-Fitting models for which the above positive variants are strongly complete. For this, note that the original definitions of Gödel-Mkrtychev or Gödel-Fitting models depend on the set of justification terms Jt and the corresponding language $\mathcal{L}_J$. Their definition can however be lifted directly to the new set of justification terms Jt_ϑ and the corresponding language $\mathcal{L}_{\vartheta J}$. Thus, if we write Gödel-Mkrtychev or Gödel-Fitting model in the following, we shall understand it as defined over the extended set of terms and formulae.

Definition 13. We call a GF-model $\langle W, R, \mathcal{E}, e\rangle$ *positive* if for all $w \in W$, all $t \in Jt_\vartheta$ and all $\phi \in \mathcal{L}_{\vartheta J}$:

$$\mathcal{E}(w, t, \phi) > 0 \text{ implies } \mathcal{E}(w, \vartheta t, \neg\neg\phi) = 1.$$

Likewise, we call a Gödel-Mkrtychev model $\langle \mathcal{E}, e\rangle$ *positive* if for any $t \in Jt_\vartheta$ and all $\phi \in \mathcal{L}_{\vartheta J}$:

$$\mathcal{E}(t, \phi) > 0 \text{ implies } \mathcal{E}(\vartheta t, \neg\neg\phi) = 1.$$

For a class C of Gödel-Mkrtychev or Gödel-Fitting models, we denote the subclass of all positive models by PC.

Remark 6.
(1) The other refinements of Gödel-Mkrtychev or Gödel-Fitting models, that is the classes introduced in Definition 3 or Definition 6, respectively, naturally carry over to the new language and we use the same jargon and notation for the corresponding model classes in the context of $\mathcal{L}_{\vartheta J}$ whenever there is no confusion.
(2) Definitions 4 and 7 naturally extend to this setting, now with $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\vartheta J}$ and where C is a class of GM- or GF-models *over* $\mathcal{L}_{\vartheta J}$, respectively. We continue to write $\models_{\mathsf{C}}$ or $\models_{\mathsf{C}\leq}$ for the respective consequence relations in these cases.

5.2. **Soundness and Completeness.** We now turn to completeness of the positive variants with respect to the provided model classes. Let $\mathcal{PGJL}_0$ be one of the aforementioned logics and CS be a constant specification for it. Let GMJL and GFJL be the classes of Gödel-Mkrtychev and Gödel-Fitting models corresponding to $\mathcal{GJL}_0$, respectively. The completeness proof is along the lines of [29] which in turn is motivated by the approach to completeness in [9] for the standard Gödel modal logics.

We begin with (strong) soundness:

Lemma 22. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\vartheta J}$*:*
(1) $\Gamma \vdash_{\mathcal{PGJL}_{CS}} \phi$ *implies* $\Gamma \models_{\mathsf{PGFJL_{CS}}\leq} \phi$*;*
(2) $\Gamma \vdash_{\mathcal{PGJL}_{CS}} \phi$ *implies* $\Gamma \models_{\mathsf{PGMJL_{CS}}\leq} \phi$*.*

Proof. At first, note that strong soundness follows directly, using the Deduction Theorem, from weak soundness (that is, the above claim instantiated with $\Gamma = \emptyset$) which in turn can be established by an induction on the length of the proof. In that way, the proof is an easy generalization of the soundness result for $\mathcal{GJL}_0$ (see [29]).

Therefore, we only show the validity of the scheme (P) in the class of positive Gödel-Fitting models. From the conditions on the evidence function, it trivially follows that the scheme is also valid in positive Gödel-Mkrtychev models.

To show validity of (P) in PGFJL_{CS}, let $\mathfrak{M} = \langle W, R, \mathcal{E}, e\rangle$ be any positive Gödel-Fitting model, $w \in W$, ϕ be a formula and t a justification term. If $\mathcal{E}(w, t, \phi) = 0$, then $|\neg\neg t : \phi|^w_{\mathfrak{M}} = 0$ and thus naturally

$$|\neg\neg t : \phi \to \vartheta t : \neg\neg\phi|^w_{\mathfrak{M}} = 1.$$

If on the other hand $\mathcal{E}(w, t, \phi) > 0$, then $\sim^2 \mathcal{E}(w, t, \phi) = 1$ and thus we have

$$\begin{aligned}|\neg\neg t : \phi|^w_{\mathfrak{M}} &= \sim^2 \mathcal{E}(w, t, \phi) \odot \sim^2 \inf\{R(w, v) \Rightarrow |\phi|^v_{\mathfrak{M}} \mid v \in W\}\\ &= \sim^2 \inf\{R(w, v) \Rightarrow |\phi|^v_{\mathfrak{M}} \mid v \in W\}.\end{aligned}$$

As $\mathfrak{M}$ is positive, $\mathcal{E}(w, t, \phi) > 0$ implies $\mathcal{E}(w, \vartheta t, \neg\neg\phi) = 1$. Hence,

$$|\vartheta t : \neg\neg\phi|^w_{\mathfrak{M}} = \inf\{R(w, v) \Rightarrow |\neg\neg\phi|^v_{\mathfrak{M}} \mid v \in W\}$$

and therefore to establish validity of (P) in this case, it suffices to show

$$\sim^2 \inf\{R(w, v) \Rightarrow |\phi|^v_{\mathfrak{M}} \mid v \in W\} \leq \inf\{R(w, v) \Rightarrow |\neg\neg\phi|^v_{\mathfrak{M}} \mid v \in W\}.$$

This is, essentially, what Caicedo and Rodriguez proved to establish the validity of the scheme (Z) (in the Gödel-Kripke models from [9], however). Suppose

$$\inf\{R(w, v) \Rightarrow |\phi|^v_{\mathfrak{M}} \mid v \in W\} > 0.$$

Then, for all $v \in W$, we have $R(w, v) = 0$ or $|\phi|^v_{\mathfrak{M}} > 0$. Therefore, we have $R(w, v) = 0$ or $|\neg\neg\phi|^v_{\mathfrak{M}} = 1$ for any $v \in W$ and thus we get $R(w, v) \Rightarrow |\neg\neg\phi|^v_{\mathfrak{M}} = 1$ for all $v \in W$. Hence, we have $\inf\{R(w, v) \Rightarrow |\neg\neg\phi|^v_{\mathfrak{M}} \mid v \in W\} = 1$. ☐

For proving the other direction of completeness, we define an auxiliary propositional language into which we convert statements containing modalities. In that way, one can reduce completeness of the Gödel justification logics to the propositional completeness results of Gödel logic. This method was employed by Caicedo and Rodriguez in [9] for approaching the completeness of the standard Gödel modal logics and adapted in [29] to the Gödel justification logics. For this, we define

$$\mathcal{L}_0(X) : \phi ::= \bot \mid \top \mid p \mid (\phi \to \phi) \mid (\phi \land \phi) \mid (\phi \lor \phi)$$

where $p \in X$ and X is a countably infinite set of variables (possibly different from Var). We may see the calculus $\mathcal{G}$ from Section 2 as being defined over $\mathcal{L}_0(X)$.

This calculus $\mathcal{G}$ is then strongly complete with respect to a $[0, 1]$-valued truth-functional semantics, defined using evaluations $v : X \to [0, 1]$, and which we will detail in the following. At first, any of these evaluations can get extended, recursively, to a function $v : \mathcal{L}_0(X) \to [0, 1]$ as follows:

- $v(\bot) := 0$; $v(\top) := 1$;
- $v(\phi \land \psi) := v(\phi) \odot v(\psi)$;
- $v(\phi \lor \psi) := v(\phi) \oplus v(\psi)$;
- $v(\phi \to \psi) := v(\phi) \Rightarrow v(\psi)$.

We denote the set of all such (extended) evaluations over the language $\mathcal{L}_0(X)$ by $\mathsf{Ev}(\mathcal{L}_0(X))$. Again, these evaluations extend further to sets of formulae $\Gamma \subseteq \mathcal{L}_0(X)$ by setting $v(\Gamma) := \inf_{\phi \in \Gamma} v(\phi)$. Also, they naturally induce an associated consequence relation, denoted here by $\models$, and which is defined by

$$\Gamma \models \phi \text{ if, and only if } \forall v \in \mathsf{Ev}(\mathcal{L}_0(X)) : v(\Gamma) = 1 \text{ implies } v(\phi) = 1$$

for $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$. The Completeness Theorem for $\mathcal{G}$ with respect to $\models$ was first proven by Dummett in [11].[4] We however refer to Hájek's proof from [19], as we use his calculus and the related t-norm semantics.

Theorem 23 (Completeness Theorem; $\mathcal{G}$ and $\models$; [19])**.** *For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0(X)$, we have $\Gamma \vdash_{\mathcal{G}} \phi$ iff $\Gamma \models \phi$.*

We now set $\mathcal{L}_0^\star := \mathcal{L}_0(Var^\star)$ where

$$Var^\star := Var \cup \{\phi_t \mid t \in Jt_\vartheta, \phi \in \mathcal{L}_{\vartheta J}\}$$

and then define the translation $\star : \mathcal{L}_{\vartheta J} \to \mathcal{L}_0^\star$, recursively, through the following clauses:

- $p \mapsto p$; $\bot \mapsto \bot$; $\top \mapsto \top$;
- $(\phi \propto \psi) \mapsto (\phi^\star \propto \psi^\star)$ for $\propto \in \{\land, \lor, \to\}$;
- $t : \phi \mapsto \phi_t$.

$\star$ extends naturally to sets of formulae Γ by $[\Gamma]^\star := \{\gamma^\star \mid \gamma \in \Gamma\}$. In analogy to the completeness proof of the standard Gödel justification logics in [29], we get the following lemma which relates the calculi $\mathcal{PGJL}_{CS}$ with $\mathcal{G}$ (over $\mathcal{L}_0^\star$) using $\star$.

Lemma 24. *For any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\vartheta J}$:*

$$\Gamma \vdash_{\mathcal{PGJL}_{CS}} \phi \textit{ if, and only if } [\Gamma]^\star \cup [Th_{\mathcal{PGJL}_{CS}}]^\star \vdash_{\mathcal{G}} \phi^\star.$$

The proof is omitted here as it amounts to two straightforward inductions on the length of the proof. See [29] for a proof of the analogous result for $\mathcal{GJL}_{CS}$.

The completeness proof culminates with the following canonical model constructions for the Gödel-Fitting and Gödel-Mkrtychev models.

Definition 14 (Canonical GF-model for $\mathcal{PGJL}_{CS}$)**.** We define $\mathfrak{M}^{c,F}(\mathcal{PGJL}_{CS}) := \langle W^c, R^c, \mathcal{E}^c, e^c\rangle$ as follows:

[4] Actually Dummett considered a different calculus and a slightly different semantics; both are equivalent to the ones presented here.

- $W^c := \{v \in \mathsf{Ev}(\mathcal{L}_0^\star) \mid v([Th_{\mathcal{PGJL}_{CS}}]^\star) = 1\}$;
- $R^c(v,w) := \begin{cases} 1 & \text{if } \forall t \in Jt_\vartheta : \forall \phi \in \mathcal{L}_{\vartheta J} : v(\phi_t) \leq w(\phi^\star); \\ 0 & \text{otherwise;} \end{cases}$
- $\mathcal{E}^c(v,t,\phi) := v(\phi_t)$;
- $e^c(v,p) := v(p)$.

Definition 15 (Canonical GM-model for $\mathcal{PGJL}_{CS}$)**.** Let $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$ such that we have $v([Th_{\mathcal{PGJL}_{CS}}]^\star) = 1$. We define $\mathfrak{M}_v^{c,M}(\mathcal{PGJL}_{CS}) := \langle \mathcal{E}^c, e^c \rangle$ as follows:

- $\mathcal{E}^c(t,\phi) := v(\phi_t)$;
- $e^c(p) := v(p)$.

Lemma 25. *Let $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$ such that $v([Th_{\mathcal{PGJL}_{CS}}]^\star) = 1$. Then the model $\mathfrak{M}_v^{c,M}(\mathcal{PGJL}_{CS})$ is a well-defined $\mathsf{PGMJL}_{\mathsf{CS}}$-model.*

Proof. We show that $\mathfrak{M}_v^{c,M}(\mathcal{PGJL}_{CS})$ is positive. For the other properties of the respective model classes associated with (F), $(!)$ or $(?)$, depending on the choice of $\mathcal{GJL}_0$, see, e.g., the proof of the analogous result in the context of the standard Gödel justification logics in [29].

Suppose $\mathcal{E}^c(t,\phi) > 0$ for some $t \in Jt_\vartheta$ and some $\phi \in \mathcal{L}_{\vartheta J}$. Per definition, we have $v(\phi_t) > 0$, i.e. $v(\neg\neg\phi_t) = 1$ and thus by the axiom scheme (P), and as $v([Th_{\mathcal{PGJL}_{CS}}]^\star) = 1$, we have $v((\neg\neg\phi)_{\vartheta t}) = 1$, i.e. per definition this yields $\mathcal{E}^c(\vartheta t, \neg\neg\phi) = 1$. □

Similarly, one obtains the following result.

Lemma 26. *$\mathfrak{M}^{c,F}(\mathcal{PGJL}_{CS})$ is a well-defined $\mathsf{PGFJL}_{\mathsf{CS}}$-model.*

We obtain the following two truth lemmas, for both the canonical Gödel-Fitting and the canonical Gödel-Mkrtychev model, respectively.

Lemma 27 (Truth Lemma; $\mathfrak{M}^{c,F}(\mathcal{PGJL}_{CS})$)**.** *Consider the canonical Gödel-Fitting model $\mathfrak{M}^{c,F}(\mathcal{PGJL}_{CS}) = \langle W^c, R^c, \mathcal{E}^c, e^c \rangle$. For any $v \in W^c$ and any $\phi \in \mathcal{L}_{\vartheta J}$:*

$$|\phi|^v_{\mathfrak{M}^{c,F}(\mathcal{PGJL}_{CS})} = v(\phi^\star).$$

Lemma 28 (Truth Lemma; $\mathfrak{M}_v^{c,M}(\mathcal{PGJL}_{CS})$)**.** *Consider $v \in \mathsf{Ev}(\mathcal{L}_0^\star)$ such that $v([Th_{\mathcal{PGJL}_{CS}}]^\star) = 1$ and $\mathfrak{M}_v^{c,M}(\mathcal{PGJL}_{CS}) = \langle \mathcal{E}^c, e^c \rangle$. For any $\phi \in \mathcal{L}_{\vartheta J}$:*

$$|\phi|_{\mathfrak{M}_v^{c,M}(\mathcal{PGJL}_{CS})} = v(\phi^\star).$$

Both lemmas can be proved with simple inductions over the structure of $\mathcal{L}_{\vartheta J}$. Alternatively, the proof from [29] for the analogous results in the context of the standard Gödel justification logics can be carried over.

Using these Truth Lemmas, one obtains the following two Completeness Theorems. We again find that, in the case of Gödel-Fitting models, only accessibility-crisp models matter for the semantic consequence (since the canonical Gödel-Fitting model is itself accessibility-crisp).

Theorem 29 (Completeness Theorem; $\mathcal{PGJL}_{CS}$ and $\mathsf{PGFJL}_{\mathsf{CS}}$)**.** *For any set $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\vartheta J}$, the following are equivalent:*

(1) $\Gamma \vdash_{\mathcal{PGJL}_{CS}} \phi$;
(2) $\Gamma \models_{\mathsf{PGFJL}_{\mathsf{CS}} \leq} \phi$;
(3) $\Gamma \models_{\mathsf{PGFJL}_{\mathsf{CS}}} \phi$;
(4) $\Gamma \models_{\mathsf{PGFJL}_{\mathsf{CS}_c}} \phi$.

Theorem 30 (Completeness Theorem; $\mathcal{PGJL}_{CS}$ and $\mathsf{PGMJL}_{\mathsf{CS}}$)**.** *For any set $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\vartheta J}$, the following are equivalent:*

(1) $\Gamma \vdash_{\mathcal{PGJL}_{CS}} \phi$;
(2) $\Gamma \models_{\mathsf{PGMJL}_{\mathsf{CS}} \leq} \phi$;
(3) $\Gamma \models_{\mathsf{PGMJL}_{\mathsf{CS}}} \phi$.

A proof is omitted but can be obtained by replicating the respective arguments from [29] for the Gödel-Mkrtychev or Gödel-Fitting models and the standard Gödel justification logics.

6. Weak standard Gödel modal logics

As we will later show, the positive Gödel justification logics are strong enough to realize the standard Gödel modal logics. In this section, we address the dual question from the beginning and introduce fragments of the standard Gödel modal logics which are realized by the standard Gödel justification logics.

For $\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$, we define $\mathcal{GML}_\Box^-$ as its reduct without the axiom scheme (Z). A first immediate observation is that, still, all instances of $\mathcal{GML}_\Box^-$ satisfy the classical Deduction Theorem:

Lemma 31 (Deduction Theorem)**.** *For any* $\Gamma \cup \{\phi, \psi\} \subseteq \mathcal{L}_\Box$*:* $\Gamma \cup \{\phi\} \vdash_{\mathcal{GML}^-_\Box} \psi$ *iff* $\Gamma \vdash_{\mathcal{GML}^-_\Box} \phi \to \psi$.

We also obtain the following lemma akin to the standard Gödel modal logics (see Lemma 6).

Lemma 32. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$*: if* $\Gamma \vdash_{\mathcal{GML}^-_\Box} \phi$*, then* $\Box\Gamma \vdash_{\mathcal{GML}^-_\Box} \Box\phi$.

A proof of this again amounts to a straightforward induction on the length of the proof.

Utilizing that the forgetful projections of the axioms of the non-positive Gödel justification logics are, in particular, also theorems of the respective weak Gödel modal logics, we obtain the following lemma:

Lemma 33. *Let* $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0\}$*,* CS *be a constant specification for* $\mathcal{GJL}_0$ *and* $\mathcal{GML}^-_\Box \in \{\mathcal{GK}^-_\Box, \mathcal{GT}^-_\Box, \mathcal{GK}4^-_\Box, \mathcal{GS}4^-_\Box\}$ *be the corresponding weak Gödel modal logic. Then for all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$*:* $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$ *implies* $\Gamma^\nu \vdash_{\mathcal{GML}^-_\Box} \phi^\nu$.

The particular instance of the above lemma associated with setting $\Gamma = \emptyset$ can again be phrased in terms of sets of theorems as follows:

Corollary 7. For $\mathcal{GJL}_{CS}$ and $\mathcal{GML}^-_\Box$ as before, we have $(Th_{\mathcal{GJL}_{CS}})^\nu \subseteq Th_{\mathcal{GML}^-_\Box}$.

One of the main objectives of this paper is now to show that the above inclusions are actually equalities, i.e. to show that these weak Gödel modal logics $\mathcal{GML}^-_\Box$ exactly represent the forgetful projection of the standard Gödel justifications logics $\mathcal{GJL}_{CS}$ (for appropriate CS).

6.1. **Semantics.** Gödel-Kripke models appear in full generality in the semantics of the standard Gödel modal logics and of course satisfy (Z). So, the question stands as of how the weak Gödel modal logics may be semantically captured. Any such semantics has to falsify the axiom scheme (Z): $\neg\neg\Box\theta \to \Box\neg\neg\theta$ (for non-provable $\neg\neg\theta$).

In the following, we present a semantics for the weak Gödel modal logics $\mathcal{GML}^-_\Box$ through natural generalizations of the Gödel-Kripke models and prove a corresponding Completeness Theorem. This semantics is again based on particular possible world models and is natural in the sense that it properly extends Gödel-Kripke models and locally respects the usual semantic evaluation of the basic propositional connectives, as Gödel-Kripke models do as well.

Definition 16. A *Quasi-Gödel-Kripke model* is a tuple $\mathfrak{M} = \langle W, R, C, e\rangle$ such that

(1) $W \neq \emptyset$, the domain of $\mathfrak{M}$ (written $\mathcal{D}(\mathfrak{M})$),
(2) $R : W \times W \to [0,1]$,
(3) $C : W \times \mathcal{L}_\Box \to [0,1]$,
(4) $e : W \times Var \to [0,1]$,

where we require that the so called *application principle* holds: for all $w \in W$ and all $\phi, \psi \in \mathcal{L}_\Box$, we have

$$C(w,\phi) \odot C(w, \phi \to \psi) \leq C(w,\psi) \quad (Appl.).$$

We call C the *controller* of $\mathfrak{M}$. The class of all Quasi-Gödel-Kripke models is denoted by QGK. Given a QGK-model $\mathfrak{M} = \langle W, R, C, e\rangle$ and a world $w \in \mathcal{D}(\mathfrak{M})$, we define the evaluation function $|\cdot|^w_\mathfrak{M} : \mathcal{L}_\Box \to [0,1]$ recursively as follows:

- $|p|^w_\mathfrak{M} := e(w,p)$ for $p \in Var$;
- $|\bot|^w_\mathfrak{M} := 0$; $|\top|^w_\mathfrak{M} := 1$;
- $|\phi \wedge \psi|^w_\mathfrak{M} := |\phi|^w_\mathfrak{M} \odot |\psi|^w_\mathfrak{M}$;
- $|\phi \to \psi|^w_\mathfrak{M} := |\phi|^w_\mathfrak{M} \Rightarrow |\psi|^w_\mathfrak{M}$;
- $|\phi \vee \psi|^w_\mathfrak{M} := |\phi|^w_\mathfrak{M} \oplus |\psi|^w_\mathfrak{M}$;
- $|\Box\phi|^w_\mathfrak{M} := C(w,\phi) \odot \inf_{v\in W}(R(w,v) \Rightarrow |\phi|^v_\mathfrak{M})$.

This extends naturally to sets of formulae Γ by setting $|\Gamma|^w_\mathfrak{M} := \inf_{\phi\in\Gamma} |\phi|^w_\mathfrak{M}$ as before. We write $(\mathfrak{M}, w) \models \phi$ if $|\phi|^w_\mathfrak{M} = 1$ and similarly for sets. Also, we write $\mathfrak{M} \models \phi$ if $(\mathfrak{M}, w) \models \phi$ for all $w \in \mathcal{D}(\mathfrak{M})$, and similarly for sets.

The application principle imposed on the controller is chosen in such a way, that every QGK-model validates the axiom scheme (K):

Lemma 34. *Let* $\mathfrak{M} = \langle W, R, C, e\rangle$ *be a* QGK*-model. For every* $\phi, \psi \in \mathcal{L}_\Box$ *and any* $w \in W$*:* $(\mathfrak{M}, w) \models \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

Proof. Let $\phi, \psi \in \mathcal{L}_\Box$ and $w \in W$ for some QGK-model $\mathfrak{M} = \langle W, R, C, e\rangle$. Now, we have for any $v \in W$:

$$\begin{aligned}
\inf_{u\in W}(R(w,u) \Rightarrow |\phi|^u_\mathfrak{M}) &\odot \inf_{u\in W}(R(w,u) \Rightarrow |\phi \to \psi|^u_\mathfrak{M}) \\
&\leq (R(w,v) \Rightarrow |\phi|^v_\mathfrak{M}) \odot (R(w,v) \Rightarrow |\phi \to \psi|^v_\mathfrak{M}) \\
&= R(w,v) \Rightarrow (|\phi|^v_\mathfrak{M} \odot |\phi \to \psi|^v_\mathfrak{M}) \\
&\leq R(w,v) \Rightarrow |\psi|^v_\mathfrak{M}.
\end{aligned}$$

By taking the infimum over v, we thus have

$$\inf_{u\in W}(R(w,u) \Rightarrow |\phi|^u_{\mathfrak{M}}) \odot \inf_{u\in W}(R(w,u) \Rightarrow |\phi \to \psi|^u_{\mathfrak{M}}) \leq \inf_{u\in W}(R(w,u) \Rightarrow |\psi|^u_{\mathfrak{M}})$$

Hence, we have

$$\begin{aligned}
&|\Box\phi|^w_{\mathfrak{M}} \odot |\Box(\phi \to \psi)|^w_{\mathfrak{M}} \\
&= \left(C(w,\phi) \odot \inf_{u\in W}(R(w,u) \Rightarrow |\phi|^u_{\mathfrak{M}})\right) \odot \\
&\qquad\qquad \left(C(w,\phi \to \psi) \odot \inf_{u\in W}(R(w,u) \Rightarrow |\phi \to \psi|^u_{\mathfrak{M}})\right) \\
&\leq C(w,\psi) \odot \inf_{u\in W}(R(w,u) \Rightarrow |\psi|^u_{\mathfrak{M}}) \\
&= |\Box\psi|^w_{\mathfrak{M}}
\end{aligned}$$

using the Application Principle for the controller which gives the result by laws of the t-norm $\odot$ and its residuum $\Rightarrow$. ☐

Over QGK-models, there are two natural notions of semantical consequence akin to those of the GK-models.

Definition 17. Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$ and C be a class of QGK-models. We then say:

1. (1) Γ entails ϕ in C, written $\Gamma \models_{\mathsf{C}\leq} \phi$, if $\forall\mathfrak{M} \in \mathsf{C} : \forall w \in \mathcal{D}(\mathfrak{M}) : |\Gamma|^w_{\mathfrak{M}} \leq |\phi|^w_{\mathfrak{M}}$;
2. (2) Γ 1-entails ϕ, written $\Gamma \models_{\mathsf{C}} \phi$, if $\forall\mathfrak{M} \in \mathsf{C} : \forall w \in \mathcal{D}(\mathfrak{M}) : (\mathfrak{M}, w) \models \Gamma$ implies $(\mathfrak{M}, w) \models \phi$.

We have to adapt QGK-models further if we want them to actually characterize any of the logics represented by $\mathcal{GML}^-_\Box$. More precisely, we at least have to avoid that the controller falsifies tautologies, as this would contradict the necessitation rule while aiming for a Completeness Theorem. It will turn out that preserving global truth in the model is already sufficient. We thus introduce the following notion.

Definition 18. We call a QGK-model $\mathfrak{M} = \langle W, R, C, e\rangle$ *regular* if for any $\phi \in \mathcal{L}_\Box$:

If $\mathfrak{M} \models \phi$, then $C(w,\phi) = 1$ for any $w \in W$.

If C is a class of QGK-models, we denote the subclass of regular models in C by RC.

Any GK-model $\mathfrak{N} = \langle W, R, e\rangle$ has a natural, semantically equivalent, RQGK-model $\mathfrak{M} := \langle W, R, C, e\rangle$ by setting $C(w,\phi) := 1$ for any $w \in W$ and any $\phi \in \mathcal{L}_\Box$. We call it the *standard conversion* of $\mathfrak{N}$.

Lemma 35. *Let $\mathfrak{M}$ be a* RQGK*-model. If $\mathfrak{M} \models \phi$, then $\mathfrak{M} \models \Box\phi$.*

Proof. Suppose $(\mathfrak{M}, w) \models \phi$, i.e. $|\phi|^w_{\mathfrak{M}} = 1$, for any $w \in \mathcal{D}(\mathfrak{M})$. Then, as $\mathfrak{M}$ is regular, we have $C(w,\phi) = 1$ and hence

$$|\Box\phi|^w_{\mathfrak{M}} = C(w,\phi) \odot \inf_{v\in W}(R(w,v) \Rightarrow |\phi|^v_{\mathfrak{M}}) = 1$$

for any $w \in W$. Thus, $\mathfrak{M} \models \Box\phi$. ☐

As a corollary, we have that $\models_{\mathsf{C}} \phi$ implies $\models_{\mathsf{C}} \Box\phi$ for any class C of RQGK-models.

In similarity to the Gödel-Kripke models, we may also introduce a range of other (more restrictive) model classes, corresponding to the various extensions of $\mathcal{GK}^-_\Box$ by the axiom schemes (T) or (4).

Definition 19. Let $\mathfrak{M} = \langle W, R, C, e\rangle$ be a QGK-model. We call $\mathfrak{M}$ a

1. (1) QGT-model if $R(w,w) = 1$ for all $w \in W$ (reflexivity),
2. (2) QGK4-model if
   - (a) $R(w,v) \odot R(v,u) \leq R(w,u)$ (min-transitivity),
   - (b) $C(w,\phi) \leq C(w,\Box\phi)$ (introspectivity),
   - (c) $C(w,\phi) \odot R(w,v) \leq C(v,\phi)$ (monotonicity),

   for all $w, v, u \in W$ and all $\phi \in \mathcal{L}_\Box$,
3. (3) QGS4-model if $\mathfrak{M}$ is both a QGT- and a QGK4-model.

Lemma 36. *For any $\phi \in \mathcal{L}_\Box$ and any* QGK*-model $\mathfrak{M}$:*

1. *(1) if $\mathfrak{M}$ is a* QGT*-model, then $\mathfrak{M} \models \Box\phi \to \phi$;*
2. *(2) if $\mathfrak{M}$ is a* QGK4*-model, then $\mathfrak{M} \models \Box\phi \to \Box\Box\phi$.*

As before, we let $\mathcal{GML}^-_\Box \in \{\mathcal{GK}^-_\Box, \mathcal{GT}^-_\Box, \mathcal{GK}4^-_\Box, \mathcal{GS}4^-_\Box\}$ and now assume RQGML to be the corresponding class of regular Quasi-Gödel-Kripke models.

It is straightforward to obtain the following soundness result.

Lemma 37. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$*:* $\Gamma \vdash_{\mathcal{GML}_\Box^-} \phi$ *implies* $\Gamma \models_{\mathsf{RQGML}_\leq} \phi$.

Proof. The proof can be carried over from the soundness proof for the standard Gödel modal logics with respect to GK-models from [9].

Note that the Necessitation Rule is valid in any class of regular models by Lemma 35 and that the modal axioms are valid in their respective classes by Lemmas 34 and 36. □

6.2. **Completeness.** We approach the Completeness Theorem in a similar way as Caicedo and Rodriguez do in [9] regarding the standard Gödel modal logics and, in particular, as before with the positive Gödel justification logics: we translate modal statements into an augmented propositional language and then use the strong completeness of propositional Gödel logic with respect to the corresponding evaluation-based semantics.

For this, let $Var^\diamond := Var \cup \{\phi_\Box \mid \phi \in \mathcal{L}_\Box\}$ and set $\mathcal{L}_0^\diamond := \mathcal{L}_0(Var^\diamond)$. We then define the translation $\diamond : \mathcal{L}_\Box \to \mathcal{L}_0^\diamond$ recursively through the following clauses:

- $p \mapsto p$ for $p \in Var$;
- $\bot \mapsto \bot$; $\top \mapsto \top$;
- $(\phi \propto \psi) \mapsto (\phi^\diamond \propto \psi^\diamond)$ for $\propto \in \{\wedge, \vee, \to\}$;
- $\Box\phi \mapsto \phi_\Box$.

Similarly to $\star$, also $\diamond$ extends to sets of formulae Γ by $[\Gamma]^\diamond := \{\gamma^\diamond \mid \gamma \in \Gamma\}$. As in [9] (and in similarity to Lemma 24), one can obtain the following lemma by induction on the length of the proof for each direction, linking $\mathcal{G}$ over $\mathcal{L}_0^\diamond$ with $\mathcal{GML}_\Box^-$ via $\diamond$.

Lemma 38. *For any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$*:* $\Gamma \vdash_{\mathcal{GML}_\Box^-} \phi$ *iff* $[\Gamma]^\diamond \cup [Th_{\mathcal{GML}_\Box^-}]^\diamond \vdash_\mathcal{G} \phi^\diamond$.

Again, the rest of the proof relies on a particular canonical model construction:

Definition 20 (Canonical RQGK-model for $\mathcal{GML}_\Box^-$)**.** We define the canonical model $\mathfrak{M}^c(\mathcal{GML}_\Box^-) := \langle W^c, R^c, C^c, e^c\rangle$ as follows:

(1) $W^c := \{v \in \mathsf{Ev}(\mathcal{L}_0^\diamond) \mid v([Th_{\mathcal{GML}_\Box^-}]^\diamond) = 1\}$;
(2) $R^c(v, w) := \begin{cases} 1 & \text{if } \forall \phi \in \mathcal{L}_\Box : v(\phi_\Box) \leq w(\phi^\diamond); \\ 0 & \text{else;} \end{cases}$
(3) $C^c(v, \phi) := v(\phi_\Box)$;
(4) $e^c(v, p) := v(p)$.

It should be noted that W^c is not empty. Through [9], we have $\not\vdash_{\mathcal{GML}_\Box} \bot$, i.e. also $\not\vdash_{\mathcal{GML}_\Box^-} \bot$. Thus by Lemma 38, we get $[Th_{\mathcal{GML}_\Box^-}]^\diamond \not\vdash_\mathcal{G} \bot$, i.e. by strong completeness of $\mathcal{G}$, there exists a $v \in \mathsf{Ev}(\mathcal{L}_0^\diamond)$ such that $v([Th_{\mathcal{GML}_\Box^-}]^\diamond) = 1$ (and naturally $v(\bot) = 0$). Hence, $v \in W^c$ for this v and thus $W^c \neq \emptyset$.

The following version of the Truth Lemma now holds for this canonical model. We give a proof of this result here as we want to emphasize that it is considerably easier to prove than the corresponding result for the canonical model of the standard Gödel modal logics in [9].

Lemma 39. *(Truth Lemma;* $\mathfrak{M}^c(\mathcal{GML}_\Box^-)$*) For any* $\phi \in \mathcal{L}_\Box$ *and any* $v \in W^c$*:* $|\phi|^v_{\mathfrak{M}^c(\mathcal{GML}_\Box^-)} = v(\phi^\diamond)$.

Proof. The proof is by induction on the structure of $\mathcal{L}_\Box$. The atomic and propositional cases are clear. Thus, suppose the claim holds for some fixed $\phi \in \mathcal{L}_\Box$ and any $w \in W^c$. For an arbitrary $v \in W^c$, it then suffices to show

$$v(\phi_\Box) = C^c(v, \phi) \odot \inf_{w \in W^c} (R^c(v, w) \Rightarrow w(\phi^\diamond))$$

as $|\phi|^w_{\mathfrak{M}^c(\mathcal{GML}_\Box^-)} = w(\phi^\diamond)$ by the induction hypothesis. As R^c is crisp and as $C^c(v, \phi) = v(\phi_\Box)$, the above is equivalent to

$$v(\phi_\Box) = v(\phi_\Box) \odot \inf\{w(\phi^\diamond) \mid w \in W^c, R^c(v, w) = 1\}.$$

By the laws of $\odot = \min$, it therefore suffices to show that

$$v(\phi_\Box) \leq \inf\{w(\phi^\diamond) \mid w \in W^c, R^c(v, w) = 1\}.$$

This, however, follows immediately from the definition of R^c: if $w \in W^c$ such that $R^c(v, w) = 1$, then per definition $v(\phi_\Box) \leq w(\phi^\diamond)$. □

Lemma 40. $\mathfrak{M}^c(\mathcal{GML}_\Box^-)$ *is a well-defined* RQGML*-model.*

Proof. First, we note that C^c fulfills the application condition: for any $\phi, \psi \in \mathcal{L}_\Box$, we have $\vdash_{\mathcal{GML}^-_\Box} \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ by axiom (K) and hence

$$(\phi \to \psi)_\Box \to (\phi_\Box \to \psi_\Box) \in [Th_{\mathcal{GML}^-_\Box}]^\diamond.$$

So, for any $v \in W^c$, as $v([Th_{\mathcal{GML}^-_\Box}]^\diamond) = 1$, we obtain

$$v((\phi \to \psi)_\Box) \odot v(\phi_\Box) \leq v(\psi_\Box)$$

by the laws of t-norms and their residua. This immediately gives the result as $C^c(v, \phi) = v(\phi_\Box)$.

Further, we note that $\mathfrak{M}^c(\mathcal{GML}^-_\Box)$ is regular. Suppose $|\phi|^v_{\mathfrak{M}^c(\mathcal{GML}^-_\Box)} = 1$ for all $v \in W^c$. By the Truth Lemma, we have

$$\forall v \in W^c : v(\phi^\diamond) = 1.$$

By definition of W^c together with the Completeness Theorem of $\mathcal{G}$, this yields $[Th_{\mathcal{GML}^-_\Box}]^\diamond \vdash_\mathcal{G} \phi^\diamond$. By Lemma 38, we have $\vdash_{\mathcal{GML}^-_\Box} \phi$ and hence $\vdash_{\mathcal{GML}^-_\Box} \Box\phi$ by necessitation. It follows that $\phi_\Box \in [Th_{\mathcal{GML}^-_\Box}]^\diamond$ and so, for all $v \in W^c$, we get $v(\phi_\Box) = 1$. By definition, $C^c(v, \phi) = 1$ for all $v \in W^c$.

Now, if $\mathcal{GML}^-_\Box$ contains the axiom scheme (T), then $\phi_\Box \to \phi^\diamond \in [Th_{\mathcal{GML}^-_\Box}]^\diamond$ for any $\phi \in \mathcal{L}_\Box$, i.e. we have $v(\phi_\Box) \leq v(\phi^\diamond)$ for any $v \in W^c$. By definition, this means $R^c(v, v) = 1$ for all $v \in W^c$.

Suppose on the other hand that $\mathcal{GML}^-_\Box$ contains the axiom scheme (4). Then, by similar reasoning as before, we have $v(\phi_\Box) \leq v((\Box\phi)_\Box)$ for any $v \in W^c$ and any formula ϕ. Thus $C^c(v, \phi) = v(\phi_\Box) \leq v((\Box\phi)_\Box) = C^c(v, \Box\phi)$. Further, for any $w, v, u \in W^c$, we get that either

(1) $R^c(v, w) \odot R^c(w, u) = 0 \leq R^c(w, u)$, or
(2) $R^c(v, w) \odot R^c(w, u) = 1$,

by crispness of R^c. For the latter, we then derive $R^c(v, w) = R^c(w, u) = 1$, i.e. we have $v(\phi_\Box) \leq w(\phi^\diamond)$ and $w(\phi_\Box) \leq u(\phi^\diamond)$ for all ϕ. Thus, for any $\phi \in \mathcal{L}_\Box$, we have

$$v(\phi_\Box) \leq v((\Box\phi)_\Box) \leq w(\phi_\Box) \leq u(\phi^\diamond)$$

for all ϕ and therefore $R^c(v, u) = 1$. Lastly, for monotonicity, suppose that we have $R^c(v, w) = 1$. Then we in particular have

$$C^c(v, \phi) = v(\phi_\Box) \leq v((\Box\phi)_\Box) \leq w(\phi_\Box) = C^c(w, \phi)$$

as above. □

Theorem 41 (Completeness Theorem; $\mathcal{GML}^-_\Box$ and RQGML)**.** *Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$. Then, the following are equivalent:*

(1) $\Gamma \vdash_{\mathcal{GML}^-_\Box} \phi$;
(2) $\Gamma \models_{\mathsf{RQGML}_\leq} \phi$;
(3) $\Gamma \models_{\mathsf{RQGML}} \phi$;
(4) $\Gamma \models_{\mathsf{RQGML}_\mathsf{c}} \phi$.

Proof. "(1) implies (2)" follows from soundness (Lemma 37) and "(2) implies (3)" as well as "(3) implies (4)" are immediate by definition. We show "(4) implies (1)" by contraposition. Suppose $\Gamma \nvdash_{\mathcal{GML}^-_\Box} \phi$. By Lemma 38, we have $[\Gamma]^\diamond \cup [Th_{\mathcal{GML}^-_\Box}]^\diamond \nvdash_\mathcal{G} \phi^\diamond$. Thus, by strong completeness of Gödel logic, there is a $v \in \mathsf{Ev}(\mathcal{L}^\diamond_0)$ such that $v([\Gamma]^\diamond) = 1$, $v \in W^c$ and $v(\phi^\diamond) < 1$. By the Truth Lemma, $|\Gamma|^v_{\mathfrak{M}^c(\mathcal{GML}^-_\Box)} = 1$ as well as $|\phi|^v_{\mathfrak{M}^c(\mathcal{GML}^-_\Box)} < 1$ and thus $\Gamma \not\models_{\mathsf{RQGML}_\mathsf{c}} \phi$ as $\mathfrak{M}^c$ is accessibility-crisp.. □

6.3. (Z) **and the weak Gödel Modal Logics.** Using this semantics and the corresponding Completeness Theorem, we can give a model-theoretical proof of the non-derivability for certain instances of the axiom scheme (Z) in the systems $\mathcal{GML}^-_\Box$. For this, consider the Quasi-Gödel-Kripke model $\mathfrak{M} = \langle \{a, b\}, R, C, e\rangle$ with $R(x, y) = 1$ iff $x = y$ for $x, y \in \{a, b\}$ as well as $e(a, p) := 1$ and $e(b, p) := 0$ for all $p \in Var$. We set $C(b, \phi) = 1$ for all $\phi \in \mathcal{L}_\Box$ and construct $C(a, \cdot)$ by recursion on $\mathcal{L}_\Box$ through:

- $C(a, \bot) := 1/2 =: C(a, p)$ for all $p \in Var$;
- $C(a, \top) := 1$;
- $C(a, \phi \wedge \psi) := C(a, \phi) \odot C(a, \psi)$;
- $C(a, \phi \vee \psi) := C(a, \phi) \oplus C(a, \psi)$;
- $C(a, \phi \to \psi) := C(a, \phi) \Rightarrow C(a, \psi)$;
- $C(a, \Box\phi) := C(a, \phi)$.

Then, we have that $\mathfrak{M}$ is a well-defined RQGS4-model. For this, it suffices to consider the following propositions:

(1) $C(a,\phi)\odot C(a,\phi\to\psi)\leq C(a,\psi)$ for all $\phi,\psi\in\mathcal{L}_\Box$;
(2) $\mathfrak{M}\models\phi$ implies $C(a,\phi)=1$ for all $\phi\in\mathcal{L}_\Box$.

This is because naturally $R(a,a)=R(b,b)=1$ by definition and R is trivially min-transitive. Also, $C(b,\cdot)$ naturally satisfies the application principle by definition. Further, both $C(a,\cdot)$ and $C(b,\cdot)$ are introspective, that is we have $C(a,\phi)\leq C(a,\Box\phi)$ and $C(b,\phi)\leq C(b,\Box\phi)$ by definition. We also immediately obtain monotonicity since $R(x,y)=1$ iff $x=y$.

For item (1), we have

$$\begin{aligned}C(a,\phi)\odot C(a,\phi\to\psi)&=C(a,\phi)\odot(C(a,\phi)\Rightarrow C(a,\psi))\\&\leq C(a,\psi)\end{aligned}$$

where the equality follows from the definition of $C(a,\cdot)$ and the inequality follows from the residuation property of $\Rightarrow$ with respect to $\odot$.

Item (2) is enough to show regularity as $C(b,\phi)=1$ holds for all ϕ by definition. To show item (2), we first prove the following intermediate claim. Here, let $\bar{+}$ be bounded addition in $[0,1]$, that is for $x,y\in[0,1]$:

$$x\,\bar{+}\,y:=\begin{cases}x+y & \text{if } x+y\leq 1;\\ 1 & \text{else.}\end{cases}$$

Then, we have:
Claim: For any $\phi\in\mathcal{L}_\Box$: $C(a,\phi)=|\phi|^b_{\mathfrak{M}}\,\bar{+}\,1/2$.
Proof: We prove the claim by syntactic induction on $\mathcal{L}_\Box$. The claim is obvious for $\bot,\top$ and $p\in Var$. For the induction step, let ϕ,ψ be such that

$$C(a,\phi)=|\phi|^b_{\mathfrak{M}}\,\bar{+}\,1/2 \text{ and } C(a,\psi)=|\psi|^b_{\mathfrak{M}}\,\bar{+}\,1/2.$$

The only interesting cases which we consider are those of $\phi\to\psi$ and $\Box\phi$. For the former, we have

$$\begin{aligned}C(a,\phi\to\psi)&=C(a,\phi)\Rightarrow C(a,\psi)\\&=\big(|\phi|^b_{\mathfrak{M}}\,\bar{+}\,1/2\big)\Rightarrow\big(|\psi|^b_{\mathfrak{M}}\,\bar{+}\,1/2\big) && \text{(by (IH))}\\&=\big(|\phi|^b_{\mathfrak{M}}\Rightarrow|\psi|^b_{\mathfrak{M}}\big)\,\bar{+}\,1/2\end{aligned}$$

where we use that $(x\,\bar{+}\,a\Rightarrow y\,\bar{+}\,a)=(x\Rightarrow y)\,\bar{+}\,a$ for any $x,y,a\in[0,1]$. To see this, suppose first that $x\leq y$. Then, clearly also $x\,\bar{+}\,a\leq y\,\bar{+}\,a$ and the equality is satisfied. Now, suppose $x>y$. Then, $(x\Rightarrow y)\,\bar{+}\,a=y\,\bar{+}\,a$ and $x\,\bar{+}\,a\geq y\,\bar{+}\,a$ where equality occurs only if $y\,\bar{+}\,a=1$. This gives the equality as well.

For the case of $\Box\phi$, we in particular have

$$|\Box\phi|^b_{\mathfrak{M}}=C(b,\phi)\odot|\phi|^b_{\mathfrak{M}}$$

as $R(b,a)=0$ and $R(b,b)=1$, and hence

$$\begin{aligned}C(a,\Box\phi)&=C(a,\phi) && \text{(by definition)}\\&=|\phi|^b_{\mathfrak{M}}\,\bar{+}\,1/2 && \text{(by (IH))}\\&=\big(C(b,\phi)\odot|\phi|^b_{\mathfrak{M}}\big)\,\bar{+}\,1/2 && \text{(as } C(b,\phi)=1)\\&=|\Box\phi|^b_{\mathfrak{M}}\,\bar{+}\,1/2.\end{aligned}$$

■

Note that we thus have that $|\phi|^b_{\mathfrak{M}}=1$ implies $C(a,\phi)=|\phi|^b_{\mathfrak{M}}\,\bar{+}\,1/2=1$. As $\mathfrak{M}\models\phi$ especially implies $|\phi|^b_{\mathfrak{M}}=1$, we obtain that $\mathfrak{M}$ is regular.

Now, we have by definition that $|\Box p|^a_{\mathfrak{M}}=C(a,p)\odot e(a,p)=1/2$ as $R(a,a)=1$ and $R(a,b)=0$, and thus $|\neg\neg\Box p|^a_{\mathfrak{M}}=1$. However, through $\neg\neg p=(p\to\bot)\to\bot$, we get that

$$\begin{aligned}C(a,\neg\neg p)&=(C(a,p)\Rightarrow C(a,\bot))\Rightarrow C(a,\bot)\\&=((1/2\Rightarrow 1/2)\Rightarrow 1/2)\\&=(1\Rightarrow 1/2)\\&=1/2\end{aligned}$$

and hence

$$\begin{aligned}|\Box\neg\neg p|^a_{\mathfrak{M}}&=C(a,\neg\neg p)\odot|\neg\neg p|^a_{\mathfrak{M}}\\&=1/2\odot 1=1/2.\end{aligned}$$

Thus, we have $|\neg\neg\Box p \to \Box\neg\neg p|^a_{\mathfrak{M}} = 1 \Rightarrow 1/2 = 1/2 < 1$ and therefore

$$\not\vdash_{\mathcal{GML}^-_\Box} \neg\neg\Box p \to \Box\neg\neg p$$

for all $p \in Var$ through Theorem 41.

6.4. **Positive Quasi-Gödel-Kripke models.** In similarity to the positive Gödel-Fitting models, we can identify positive Quasi-Gödel-Kripke models $\mathfrak{M} = \langle W, R, C, e\rangle$ as those satisfying

$$C(w, \phi) > 0 \text{ implies } C(w, \neg\neg\phi) = 1$$

for all $w \in W$ and $\phi \in \mathcal{L}_\Box$. For a class of QGK-models C, we denote the subclass of all positive models in C by PC.

Lemma 42. *Let $\mathfrak{M} \in \mathsf{PQGK}$ and let $w \in \mathcal{D}(\mathfrak{M})$. Then, for all $\phi \in \mathcal{L}_\Box$, we have $(\mathfrak{M}, w) \models \neg\neg\Box\phi \to \Box\neg\neg\phi$.*

Proof. The proof of

$$\sim^2 \inf\{R(w, v) \Rightarrow |\phi|^v_{\mathfrak{M}} \mid v \in W\} \leq \inf\{R(w, v) \Rightarrow |\neg\neg\phi|^v_{\mathfrak{M}} \mid v \in W\}$$

is similar to the one presented in the proof of Lemma 22 for an analogous statement regarding the axiom scheme (P) in the context of the positive Gödel justification logics together with (positive) Gödel-Fitting models which in turn is a replication of the proof of validity for the axiom scheme (Z) over Gödel-Kripke models in Caicedo's and Rodriguez' paper [9]. Further, we immediately have

$$\sim^2 C(w, \phi) \leq C(w, \neg\neg\phi)$$

in positive models as if $C(w, \phi) > 0$, then $C(w, \neg\neg\phi) = 1$ by definition of positivity. Thus, using the equality $\sim^2 (x \odot y) = \sim^2 x \odot \sim^2 y$, we obtain the result by monotonicity of $\odot$. $\Box$

Indeed, we find that the positive regular Quasi-Gödel-Kripke models, semantically, exactly classify the standard Gödel modal logics. More precisely, we obtain the following theorem:

Theorem 43 (Completeness Theorem; $\mathcal{GML}_\Box$ and PRQGML)**.** *Let*

$$\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}$$

and let PRQGML be the corresponding class of positive regular Quasi-Gödel-Kripke models. Then, for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_\Box$, the following are equivalent:

(1) $\Gamma \vdash_{\mathcal{GML}_\Box} \phi$;
(2) $\Gamma \models_{\mathsf{PRQGML}_\leq} \phi$;
(3) $\Gamma \models_{\mathsf{PRQGML}} \phi$;
(4) $\Gamma \models_{\mathsf{PRQGML}_\mathsf{c}} \phi$.

Proof. "(1) implies (2)" follows from Lemma 42 in the same vein as the other soundness results (see e.g. Lemma 37). "(2) implies (3)" and "(3) implies (4)" are natural consequences of the definition of the consequence relation and the various classes.

For "(4) implies (1)", suppose $\Gamma \not\vdash_{\mathcal{GML}_\Box} \phi$. By Theorem 8 there is an accessibility-crisp GML-model $\mathfrak{M}$ and a world $w \in \mathcal{D}(\mathfrak{M})$ such that $(\mathfrak{M}, w) \models \Gamma$ but $(\mathfrak{M}, w) \not\models \phi$. Taking this $\mathfrak{M} = \langle W, R, e\rangle$, construct its standard conversion $\mathfrak{N} = \langle W, R, C, e\rangle$ with $C(w, \phi) = 1$ for all w, ϕ. Then, it is straightforward to verify that $\mathfrak{N} \in \mathsf{PRQGML}_\mathsf{c}$ and that $|\phi|^w_{\mathfrak{M}} = |\phi|^w_{\mathfrak{N}}$ for all $w \in W$ and all $\phi \in \mathcal{L}_\Box$. The latter follows from a straightforward induction on the structure of $\mathcal{L}_\Box$. Thus, we have $(\mathfrak{N}, w) \models \Gamma$ but $(\mathfrak{N}, w) \not\models \phi$ and so $\Gamma \not\models_{\mathsf{PRQGML}_\mathsf{c}} \phi$. $\Box$

7. Hypersequent calculi

Most constructive proofs of realization results in the context of classical justification logics (and beyond) rely on appropriate cut-free structural calculi for the modal logic in question, like cut-free sequent calculi or analytic tableau calculi and their siblings (see [4, 25] for comprehensive treatments of constructive realization in the classical case).

For propositional Gödel logics, there is no sequent calculi formulation available. The most common structural proof-theory approach (see [6]) uses so-called hypersequent calculi which generalize sequent calculi by allowing to work on multiple sequents in parallel and have these sequents "exchange information". These hypersequent calculi were introduced in [5, 30], noting especially Avron's [5] where these calculi were applied, in particular, to propositional Gödel logic.

In [27], Metcalfe and Olivetti provided a hypersequent calculus for the standard Gödel modal logic $\mathcal{GK}_\Box$ from [9] together with a corresponding Cut-Elimination Theorem and which extends Avron's hypersequent calculus for $[0, 1]$-valued propositional Gödel logic from [5]. In the following, we now introduce fragments of the calculus of Metcalfe and Olivetti and some extensions, show cut-elimination, and that these calculi characterize the previously introduced Hilbert-style proof systems represented by $\mathcal{GML}_\Box^-$ and $\mathcal{GML}_\Box$.

We refer to an ordered pair (Γ, Δ) of finite multisets of formulae, where Δ contains at most one formula, as a *sequent* and write $\Gamma \rhd \Delta$. We use $\rhd$ instead of the common $\Rightarrow$ as a sequent delimiter to avoid confusion with the semantical Gödel implication which is denoted by $\Rightarrow$ in this paper. We also write $[\phi_1, \dots, \phi_n]$ (or even "$\phi_1, \dots, \phi_n$" in the context of $\rhd$) for denoting the multiset containing (the not necessarily distinct) formulae $\phi_1, \dots, \phi_n$ and write $[\phi]^n$ for the multiset containing ϕ exactly n times. For Γ or Δ being empty, we also write

$$\rhd \Delta \text{ or } \Gamma \rhd,$$

respectively, and $\Gamma \rhd \phi$ for $\Gamma \rhd [\phi]$. With "Γ, Δ", we denote the multiset-union of Γ and Δ.

We call a multiset of finitely many sequents $\Gamma_i \rhd \Delta_i$ (for $i = 1, \dots, n$) a *hypersequent* and write

$$\Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_n \rhd \Delta_n \text{ or } [\Gamma_i \rhd \Delta_i]_{i=1}^n$$

as representations for that multiset.

For a multiset of formulae Γ, we write $\bigwedge \Gamma$ or $\bigvee \Gamma$ for the conjunction or disjunction of all members of Γ, respectively, including repetitions. We set $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bot$.

With every hypersequent G, we then associate its canonical interpretation $\mathcal{I}(G)$ into the language of its formulae which we define for a single sequent $\Gamma \rhd \Delta$ as

$$\mathcal{I}(\Gamma \rhd \Delta) := \bigwedge \Gamma \to \bigvee \Delta$$

and for a hypersequent $\Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_n \rhd \Delta_n$ by

$$\mathcal{I}(\Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_n \rhd \Delta_n) := \bigvee_{i=1}^{n} \mathcal{I}(\Gamma_i \rhd \Delta_i).$$

The range of constituting rules of the various hypersequent calculi can be seen in Fig. 7. We refer with

1. $\mathcal{HGK}_\Box$ to all initial, structural and logical rules with the modal rule $(\Box)$,
2. $\mathcal{HGT}_\Box$ to $\mathcal{HGK}_\Box$ extended with the rule $(\Box \rhd)$,
3. $\mathcal{HGK}4_\Box$ to all initial, structural and logical rules with the modal rule $(\rhd \Box)_1$,
4. $\mathcal{HGS}4_\Box$ to all initial, structural and logical rules with the modal rules $(\Box \rhd)$ and $(\rhd \Box)_2$.

For $\mathcal{HGML}_\Box$ being one of the above systems, we refer with $\mathcal{HGML}_\Box^-$ to the version where the rule $(\Box)$ is replaced by $(\Box^-)$, the rule $(\rhd \Box)_1$ by $(\rhd \Box^-)_1$ and the rule $(\rhd \Box)_2$ by $(\rhd \Box^-)_2$, respectively, if any of them are contained in $\mathcal{HGML}_\Box$. Given a hypersequent G, a hypersequent calculus $\mathcal{S}_\Box$ and a tree of hypersequents γ, we write $\gamma \vdash_{\mathcal{S}_\Box} G$ if γ is a proof of G in $\mathcal{S}_\Box$ (as usually defined for (hyper-)sequent calculi) and $\vdash_{\mathcal{S}_\Box} G$ if there is such a proof.

As said before, the particular system $\mathcal{HGK}_\Box$ was (in some way) already considered by Metcalfe and Olivetti in [27]. Originally however, Metcalfe and Olivetti considered a hypersequent calculus where the rules (ID^w), $(\bot\rhd^\mathsf{w})$ and $(\rhd\top^\mathsf{w})$ were replaced by the following *strong versions*:

$$\frac{}{G \mid \phi \rhd \phi}\ (\mathsf{ID}); \qquad \frac{}{G \mid \Gamma, \bot \rhd \Delta}\ (\bot\rhd); \qquad \frac{}{G \mid \Gamma \rhd \top}\ (\rhd\top).$$

However, these rules are derivable using the weak versions in the calculi presented here. Some useful derived rules (following [27]) are the following regarding the introduction and removal of negation (as a derived connective):

$$\frac{G \mid \Gamma \rhd \phi}{G \mid \Gamma, \neg\phi \rhd}\ (\neg\rhd); \qquad \frac{G \mid \Gamma, \phi \rhd}{G \mid \Gamma \rhd \neg\phi}\ (\rhd\neg).$$

It is quite obvious that the rule $(\Box \rhd)$ suffices (in the context of the initial, structural and logical rules) to prove the hypersequent variant of the axiom scheme (T) by

$$\dfrac{\dfrac{\dfrac{}{\phi \rhd \phi}\ (\mathsf{ID})}{\Box\phi \rhd \phi}\ (\Box\rhd)}{\rhd \Box\phi \to \phi.}\ (\rhd \to)$$

A similar argument shows this for the axiom scheme (4) and the rule $(\rhd \Box^-)_1$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\phi \rhd \Box\phi}\ (\mathsf{ID})}{\Box\phi, \phi \rhd \Box\phi}\ (\mathsf{WL})}{\Box\phi \rhd \Box\Box\phi}\ (\rhd \Box^-)_1}{\rhd \Box\phi \to \Box\Box\phi.}\ (\rhd \to)$$

Note that in general the rule $(\Box^-)$ is derivable in $\mathcal{HGK}_\Box$ (and its extensions) by

Initial Hypersequents

$$\frac{}{p \rhd p}\ (\mathsf{ID}^{\mathsf{w}}),\ p \in Var \qquad \frac{}{\bot \rhd}\ (\bot\rhd^{\mathsf{w}}) \qquad \frac{}{\rhd \top}\ (\rhd\top^{\mathsf{w}})$$

Structural Rules

$$\frac{G}{G \mid H}\ (\mathsf{EW}) \qquad \frac{G \mid H \mid H}{G \mid H}\ (\mathsf{EC}) \qquad \frac{G \mid \Gamma_1, \Pi_1 \rhd \Delta_1 \quad G \mid \Gamma_2, \Pi_2 \rhd \Delta_2}{G \mid \Gamma_1, \Gamma_2 \rhd \Delta_1 \mid \Pi_1, \Pi_2 \rhd \Delta_2}\ (\mathsf{COM})$$

$$\frac{G \mid \Gamma \rhd \Delta}{G \mid \Gamma, \phi \rhd \Delta}\ (\mathsf{WL}) \qquad \frac{G \mid \Gamma \rhd}{G \mid \Gamma \rhd \phi}\ (\mathsf{WR}) \qquad \frac{G \mid \Gamma, \phi, \phi \rhd \Delta}{G \mid \Gamma, \phi \rhd \Delta}\ (\mathsf{CL})$$

Logical Rules

$$\frac{G \mid \Gamma_1 \rhd \phi \quad G \mid \Gamma_2, \psi \rhd \Delta}{G \mid \Gamma_1, \Gamma_2, \phi \to \psi \rhd \Delta}\ (\to\rhd) \qquad \frac{G \mid \Gamma, \phi \rhd \psi}{G \mid \Gamma \rhd \phi \to \psi}\ (\rhd\to)$$

$$\frac{G \mid \Gamma, \phi \rhd \Delta}{G \mid \Gamma, \phi \wedge \psi \rhd \Delta}\ (\wedge\rhd)_1 \qquad \frac{G \mid \Gamma, \psi \rhd \Delta}{G \mid \Gamma, \phi \wedge \psi \rhd \Delta}\ (\wedge\rhd)_2$$

$$\frac{G \mid \Gamma \rhd \phi \quad G \mid \Gamma \rhd \psi}{G \mid \Gamma \rhd \phi \wedge \psi}\ (\rhd\wedge) \qquad \frac{G \mid \Gamma, \phi \rhd \Delta \quad G \mid \Gamma, \psi \rhd \Delta}{G \mid \Gamma, \phi \vee \psi \rhd \Delta}\ (\vee\rhd)$$

$$\frac{G \mid \Gamma \rhd \phi}{G \mid \Gamma \rhd \phi \vee \psi}\ (\rhd\vee)_1 \qquad \frac{G \mid \Gamma \rhd \psi}{G \mid \Gamma \rhd \phi \vee \psi}\ (\rhd\vee)_2$$

Modal rules

$$\frac{\Gamma \rhd \phi}{\Box\Gamma \rhd \Box\phi}\ (\Box^-) \qquad \frac{\Pi \rhd \mid \Gamma \rhd \phi}{\Box\Pi \rhd \mid \Box\Gamma \rhd \Box\phi}\ (\Box) \qquad \frac{G \mid \phi, \Gamma \rhd \Delta}{G \mid \Box\phi, \Gamma \rhd \Delta}\ (\Box\rhd)$$

$$\frac{\Gamma, \Box\Gamma \rhd \phi}{\Box\Gamma \rhd \Box\phi}\ (\rhd\Box^-)_1 \qquad \frac{\Box\Gamma \rhd \phi}{\Box\Gamma \rhd \Box\phi}\ (\rhd\Box^-)_2$$

$$\frac{\Pi, \Box\Pi \rhd \mid \Gamma, \Box\Gamma \rhd \phi}{\Box\Pi \rhd \mid \Box\Gamma \rhd \Box\phi}\ (\rhd\Box)_1 \qquad \frac{\Box\Pi \rhd \mid \Box\Gamma \rhd \phi}{\Box\Pi \rhd \mid \Box\Gamma \rhd \Box\phi}\ (\rhd\Box)_2$$

Cut rule

$$\frac{G \mid \Gamma_1, \phi \rhd \Delta \quad G \mid \Gamma_2 \rhd \phi}{G \mid \Gamma_1, \Gamma_2 \rhd \Delta}\ (\mathsf{CUT})$$

FIGURE 1. The various hypersequent rules.

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \rhd \phi}{\Gamma \rhd \mid \Gamma \rhd \phi}\ (\mathsf{EW})}{\Box\Gamma \rhd \mid \Box\Gamma \rhd \Box\phi}\ (\Box)}{\Box\Gamma \rhd \Box\phi \mid \Box\Gamma \rhd \Box\phi}\ (\mathsf{WR})}{\Box\Gamma \rhd \Box\phi.}\ (\mathsf{EC}).$$

Also note that $(\rhd\Box)_1, (\rhd\Box^-)_1$ suffice to derive $(\Box), (\Box^-)$, respectively, and that $(\rhd\Box)_2, (\rhd\Box^-)_2$, in the presence of $(\Box\rhd)$, suffice to derive $(\rhd\Box)_1, (\rhd\Box^-)_1$, respectively. Given a multiset of formulae Γ and some proof calculus $\mathcal{S}$ over the same language, we write $\Gamma \vdash_{\mathcal{S}} \phi$ if $\hat{\Gamma} \vdash_{\mathcal{S}} \phi$ where $\hat{\Gamma}$ is the set corresponding to Γ.

Theorem 44 (Weak Completeness Theorem; $\mathcal{GML}^{\pm}_{\Box}$ and $\mathcal{HGML}^{\pm}_{\Box} + (\mathsf{CUT})$)**.** *Let*

$$\mathcal{GML}^{\pm}_{\Box} \in \{\mathcal{GK}^{-}_{\Box}, \mathcal{GT}^{-}_{\Box}, \mathcal{GK}4^{-}_{\Box}, \mathcal{GS}4^{-}_{\Box}, \mathcal{GK}_{\Box}, \mathcal{GT}_{\Box}, \mathcal{GK}4_{\Box}, \mathcal{GS}4_{\Box}\}.$$

Then, for every $\phi \in \mathcal{L}_{\Box}$:

$$\vdash_{\mathcal{GML}^{\pm}_{\Box}} \phi \text{ iff } \vdash_{\mathcal{HGML}^{\pm}_{\Box}+(\mathsf{CUT})} \rhd \phi.$$

Proof. For the direction from left to right, we show the claim by induction on the length of the proof. Naturally, for every axiom instance ϕ of $\mathcal{GML}^{\pm}_{\Box}$, $\rhd\phi$ is derivable in $\mathcal{HGML}^{\pm}_{\Box} + (\mathsf{CUT})$. This is clear for the propositional axioms. For the modal axioms $(T), (4)$, this was indicated above and, naturally, (K) can be derived using $(\Box^-)$ and consequently also using $(\Box)$. The axiom scheme (Z) can be derived using $(\Box)$ as shown in [27].

Now, suppose the formula ϕ was obtained by modus ponens with $\vdash_{\mathcal{GML}^{\pm}_{\Box}} \psi$ and $\vdash_{\mathcal{GML}^{\pm}_{\Box}} \psi \to \phi$. By induction hypothesis, we get $\vdash_{\mathcal{HGML}^{\pm}_{\Box}+(\mathsf{CUT})} \rhd\psi$ as well as $\vdash_{\mathcal{HGML}^{\pm}_{\Box}+(\mathsf{CUT})} \rhd\psi \to \phi$ and therefore we obtain

$$\dfrac{\dfrac{\dfrac{}{\psi \rhd \psi}\,(\mathsf{ID}) \quad \dfrac{}{\phi \rhd \phi}\,(\mathsf{ID})}{\psi, \psi \to \phi \rhd \phi}\,(\to \rhd) \quad \dfrac{}{\rhd \psi \to \phi}\,(\text{Assm.})}{\dfrac{\psi \rhd \phi}{}}\,(\mathsf{CUT}) \quad \dfrac{}{\rhd \psi}\,(\text{Assm.})$$

$$\dfrac{\psi \rhd \phi \qquad \rhd \psi}{\rhd \phi}\,(\mathsf{CUT})$$

as a proof of $\rhd \phi$ in $\mathcal{HGML}^{\pm}_{\Box} + (\mathsf{CUT})$.

For the direction from right to left, we actually show that $\vdash_{\mathcal{HGML}^{\pm}_{\Box}+(\mathsf{CUT})} G$ implies $\vdash_{\mathcal{GML}^{\pm}_{\Box}} \mathcal{I}(G)$ for any hypersequent G. This is again proved by induction on the length of the proof. Note that the translations of the initial hypersequents are naturally theorems of $\mathcal{GML}^{\pm}_{\Box}$. See [27] for the validity (or admissibility) of the structural and logical rules.

The admissibility of $(\Box)$ with respect to the corresponding calculi is shown in Lemma 7 (see also Remark 3), the admissibility of $(\Box^-)$ in Lemma 32 and Lemma 6. The admissibility of the rules $(\rhd\Box)_1, (\rhd\Box)_2, (\rhd\Box^-)_1$ and $(\rhd\Box^-)_2$ follows from a straightforward use of the axiom schemes (T) and (4) together with Lemma 7 and Lemma 32 ☐

We now present a Cut-Elimination Theorem for the various hypersequent calculi introduced before. The method of the proof is due to Avron [5] and is used in [27] to prove cut-elimination of $\mathcal{HGK}_{\Box}$. A survey about this (in the non-modal propositional case) can be found in [6]. We don't get into the issues of dealing with (internal or external) contraction and the consequent introduction of a hypersequent version of the multicut (or mix) rule of Gentzen. Instead, we focus on the modal rules.

For this, note that the following multi-rules of $(\Box), (\rhd\Box)_1$ and $(\rhd\Box)_2$ are derivable in the respective systems (where $m \in \mathbb{N} \setminus \{0\}$):

$$\dfrac{\Pi_1 \rhd \mid \cdots \mid \Pi_m \rhd \mid \Gamma \rhd \phi}{\Box\Pi_1 \rhd \mid \cdots \mid \Box\Pi_m \rhd \mid \Box\Gamma \rhd \Box\phi}\,\mathsf{m}(\Box)$$

$$\dfrac{\Pi_1, \Box\Pi_1 \rhd \mid \cdots \mid \Pi_m, \Box\Pi_m \rhd \mid \Gamma, \Box\Gamma \rhd \phi}{\Box\Pi_1 \rhd \mid \cdots \mid \Box\Pi_m \rhd \mid \Box\Gamma \rhd \Box\phi}\,\mathsf{m}(\rhd\Box)_1$$

$$\dfrac{\Box\Pi_1 \rhd \mid \cdots \mid \Box\Pi_m \rhd \mid \Box\Gamma \rhd \phi}{\Box\Pi_1 \rhd \mid \cdots \mid \Box\Pi_m \rhd \mid \Box\Gamma \rhd \Box\phi}\,\mathsf{m}(\rhd\Box)_2$$

An example derivation of $2(\rhd\Box)_1$ goes as follows: we obtain

$$\dfrac{\dfrac{\dfrac{\Pi_1, \Box\Pi_1 \rhd \mid \Pi_2, \Box\Pi_2 \rhd \mid \Gamma, \Box\Gamma \rhd \phi}{\Pi_1, \Pi_2, \Box\Pi_1, \Box\Pi_2 \rhd \mid \Pi_1, \Pi_2, \Box\Pi_1, \Box\Pi_2 \rhd \mid \Gamma, \Box\Gamma \rhd \phi}\,(\mathsf{WL})^*}{\Pi_1, \Pi_2, \Box\Pi_1, \Box\Pi_2 \rhd \mid \Gamma, \Box\Gamma \rhd \phi}\,(\mathsf{EC})}{\Box\Pi_1, \Box\Pi_2 \rhd \mid \Box\Gamma \rhd \Box\phi}\,(\rhd\Box)_1$$

from the assumption.[5] Using this twice, for both arguments of (COM), we get:

$$\dfrac{\dfrac{\Box\Pi_1, \Box\Pi_2 \rhd \mid \Box\Gamma \rhd \Box\phi \qquad \Box\Pi_1, \Box\Pi_2 \rhd \mid \Box\Gamma \rhd \Box\phi}{\Box\Pi_1, \Box\Pi_1 \rhd \mid \Box\Pi_2, \Box\Pi_2 \rhd \mid \Box\Gamma \rhd \Box\phi}\,(\mathsf{COM})}{\Box\Pi_1 \rhd \mid \Box\Pi_2 \rhd \mid \Box\Gamma \rhd \Box\phi.}\,(\mathsf{CL})^*$$

For a similar derivation of $2(\Box)$ see [27]; a derivation of $2(\rhd\Box)_2$ may be obtained in the same way. The general instances $\mathsf{m}(\Box), \mathsf{m}(\rhd\Box)_1$ and $\mathsf{m}(\rhd\Box)_2$ can be derived by natural generalizations.

Theorem 45 (Cut Elimination)**.** *Let*

$$\mathcal{HGML}^{\pm}_{\Box} \in \{\mathcal{HGK}^-_{\Box}, \mathcal{HGT}^-_{\Box}, \mathcal{HGK}4^-_{\Box}, \mathcal{HGS}4^-_{\Box}, \mathcal{HGK}_{\Box}, \mathcal{HGT}_{\Box}, \mathcal{HGK}4_{\Box}, \mathcal{HGS}4_{\Box}\}.$$

Then, for every hypersequent G:

$$\vdash_{\mathcal{HGML}^{\pm}_{\Box}+(\mathsf{CUT})} G \text{ iff } \vdash_{\mathcal{HGML}^{\pm}_{\Box}} G.$$

Proof. We sketch the proof. The general structure is similar to those of the proofs of cut-elimination from [6, 27]. Similarly, it suffices to prove the following claim where we write $\Gamma = \Gamma_1, \dots, \Gamma_n$:

$$(*)\begin{cases} \text{If } \vdash_{\mathcal{HGML}^{\pm}_{\Box}} G := G_1 \mid \Gamma_1 \rhd \phi \mid \cdots \mid \Gamma_n \rhd \phi \text{ and} \\ \quad \vdash_{\mathcal{HGML}^{\pm}_{\Box}} H := H_1 \mid \Sigma_1, [\phi]^{n_1} \rhd \psi_1 \mid \cdots \mid \Sigma_k, [\phi]^{n_k} \rhd \psi_k, \\ \text{then } \vdash_{\mathcal{HGML}^{\pm}_{\Box}} G_1 \mid H_1 \mid \Gamma, \Sigma_1 \rhd \psi_1 \mid \cdots \mid \Gamma, \Sigma_k \rhd \psi_k. \end{cases}$$

[5]Here, and in the following, we write $(R)^*$ for multiple applications of a rule (R) in the proof tree.

It suffices to establish $(*)$ for the cut-elimination theorem: if we have $\vdash_{\mathcal{HGML}^{\pm}_{\Box}+(\mathsf{CUT})} I$ for some hypersequent I with exactly one application of (CUT), then the premises of the cut are provable in $\mathcal{HGML}^{\pm}_{\Box}$. Therefore, $(*)$ applies and the (CUT)-application can be replaced by the (CUT)-free proof from the conclusion of $(*)$. The full statement now follows by induction over the number of cuts.[6]

Let γ, η be proofs for G and H respectively. The claim $(*)$ is proven by induction on the lexicographically ordered pair $(c(\phi), |\gamma| + |\eta|)$ where $c(\phi)$ is the complexity of ϕ (see Section 2.2) and $|\gamma|$ is the height of the derivation γ as a tree (similarly for $|\eta|$). We refer to ϕ as the *cut-formula*.

There, it is sufficient to consider the following cases:

(1) G or H are initial hypersequents;
(2) γ or η end by an application of a structural rule;
(3) γ and η end in a logical (or modal) rule.

This division is sufficient as all combinations of initial, structural and logical (or modal) rules in which γ and η can end are covered.

Case (1) is almost trivial as the cut formula is either $p \in Var$ or $\bot$ or $\top$ as one (or both) of G, H have been obtain by an initial rule. For case (2) see e.g. [27] or [6] for relevant instances where the corresponding arguments translate directly to the systems considered here. See especially [6] regarding arguments for (COM) or (EC) and [27] for arguments regarding (COM) in combination with $(\Box)$.

We thus only consider interesting instances of item (3) involving the rules $(\Box)$, $(\Box \rhd)$, $(\rhd \Box)_1$ and $(\rhd \Box)_2$. Cases involving the weak versions $(\Box^-), (\rhd \Box^-)_1$ and $(\rhd \Box^-)_2$ can be seen as special cases of those.

Of course there are a multitude of cases to consider, but the harder (or more interesting) cases which we want to focus on, involving these modal rules, are:[7]

(i) γ and η end with $(\rhd \Box)_1$;
(ii) γ and η end with $(\rhd \Box)_2$;
(iii) γ ends with $(\rhd \Box)_2$ and η ends with $(\Box \rhd)$;
(iv) γ ends with $(\Box)$ and η ends with $(\Box \rhd)$;
(v) γ and η end with $(\Box)$.

Case (v) was discussed in [27] and the same argument applies here. Case (ii) is very similar to case (i) and in the same way, case (iv) is very similar to case (iii). Thus, we only discuss case (i) and case (iii) explicitly as a straightforward adaption of the arguments for (i),(iii) also solves (ii),(iv), respectively.

(i) γ and η end with $(\rhd \Box)_1$. By assumption, we have that γ ends in

$$\dfrac{\dfrac{\vdots}{\Pi_1, \Box\Pi_1 \rhd \mid \Gamma, \Box\Gamma \rhd \phi}}{\Box\Pi_1 \rhd \mid \Box\Gamma \rhd \Box\phi}\,(\rhd \Box)_1$$

and that η ends in

$$\dfrac{\dfrac{\vdots}{\Pi_2, \Box\Pi_2, [\phi]^n, [\Box\phi]^n \rhd \mid \Sigma, \Box\Sigma, [\phi]^m, [\Box\phi]^m \rhd \psi}}{\Box\Pi_2, [\Box\phi]^n \rhd \mid \Box\Sigma, [\Box\phi]^m \rhd \Box\psi.}\,(\rhd \Box)_1$$

Applying the induction hypothesis to $\Box\phi$ and the last line of γ together with the second-to-last line of η (as the sum of the heights of those proof-trees is smaller than $|\gamma| + |\eta|$), we obtain

$$\dfrac{\vdots\ \text{cut free}}{\Box\Pi_1 \rhd \mid \Box\Gamma, \Pi_2, \Box\Pi_2, [\phi]^n \rhd \mid \Box\Gamma, \Sigma, \Box\Sigma, [\phi]^m \rhd \psi.}$$

Using this and the second-to-last line of γ in the induction hypothesis (as $|\phi| < |\Box\phi|$), we then have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \text{cut free}}{\Pi_1, \Box\Pi_1 \rhd \mid \Box\Pi_1 \rhd \mid \Box\Gamma, \Pi_2, \Box\Pi_2, \Gamma, \Box\Gamma \rhd \mid \Box\Gamma, \Sigma, \Box\Sigma, \Gamma, \Box\Gamma \rhd \psi}}{\Pi_1, \Box\Pi_1 \rhd \mid \Box\Pi_1 \rhd \mid \Box\Gamma, \Pi_2, \Box\Pi_2, \Gamma \rhd \mid \Box\Gamma, \Sigma, \Box\Sigma, \Gamma \rhd \psi}\,(\mathsf{CL})^*}{\Pi_1, \Box\Pi_1 \rhd \mid \Pi_1, \Box\Pi_1 \rhd \mid \Box\Gamma, \Pi_2, \Box\Pi_2, \Gamma \rhd \mid \Box\Gamma, \Sigma, \Box\Sigma, \Gamma \rhd \psi}\,(\mathsf{WL})^*}{\Pi_1, \Box\Pi_1 \rhd \mid \Box\Gamma, \Pi_2, \Box\Pi_2, \Gamma \rhd \mid \Box\Gamma, \Sigma, \Box\Sigma, \Gamma \rhd \psi}\,(\mathsf{EC})}{\Box\Pi_1 \rhd \mid \Box\Gamma, \Box\Pi_2 \rhd \mid \Box\Gamma, \Box\Sigma \rhd \Box\psi.}\,2(\rhd \Box)_1$$

[6]Of course, the statement $(*)$ contains multiple cut-sequents with multiple cuts inside them which are eliminated at the same time. For the simple statement of cut-elimination in the theorem we prove, this is of course oversaturated but needed in order to deal with contraction. Note our comment after the previous weak Completeness Theorem on that we don't get into the surrounding details.

[7]In all cases, $\Box\phi$ is the cut-formula.

(iii) γ ends with $(\rhd\Box)_2$ and η ends with $(\Box\rhd)$. By assumption, γ ends in

$$\dfrac{\dfrac{\vdots}{\Box\Pi\rhd \mid \Box\Gamma\rhd\phi}}{\Box\Pi\rhd \mid \Box\Gamma\rhd\Box\phi}\ (\rhd\Box)_2$$

and η ends in

$$\dfrac{\dfrac{\vdots}{G \mid \Sigma, [\Box\phi]^n, \phi\rhd\psi}}{G \mid \Sigma, [\Box\phi]^{n+1}\rhd\psi.}\ (\Box\rhd)$$

By the induction hypothesis applied to $\Box\phi$ and the last line of γ together with the second to last line of η, we obtain:

$$\dfrac{\vdots\ \text{cut free}}{G \mid \Box\Pi\rhd \mid \Box\Gamma, \Sigma, \phi\rhd\psi.}$$

Applying the induction hypothesis again to ϕ from the above and the second-to-last line of γ, we derive:

$$\dfrac{\dfrac{\dfrac{\vdots\ \text{cut free}}{G \mid \Box\Pi\rhd \mid \Box\Pi\rhd \mid \Box\Gamma, \Box\Gamma, \Sigma\rhd\psi}}{G \mid \Box\Pi\rhd \mid \Box\Gamma, \Box\Gamma, \Sigma\rhd\psi}\ (\mathsf{EC})}{G \mid \Box\Pi\rhd \mid \Box\Gamma, \Sigma\rhd\psi.}\ (\mathsf{CL})^*$$

□

8. Annotations and realizations

To provide a formal footing for the Realization Theorem and its proof, we introduce *annotated modal formulae*, following e.g. [7, 8, 14].

Definition 21. We define the language $\mathcal{L}'_\Box$ by

$$\mathcal{L}'_\Box : \phi' ::= \bot \mid \top \mid p \mid (\phi' \to \phi') \mid (\phi' \land \phi') \mid (\phi' \lor \phi') \mid \Box_i\phi'$$

where $p \in Var$ and $i \in \mathbb{N}$.

The function sf naturally extends to $\mathcal{L}'_\Box$. There is a natural projection from $\mathcal{L}'_\Box$ to $\mathcal{L}_\Box$ by just mapping every $\Box_i$ to $\Box$, preserving the propositional part of the formula as it is. We call this projection $\bullet$. An annotated formula $\phi' \in \mathcal{L}'_\Box$ is called an *annotation* of a formula $\psi \in \mathcal{L}_\Box$ if $(\phi')^\bullet = \psi$.

A central notion in the context of realizations, in particular for constructing realizations later on, is the sign of a modal operator. In the following, let $\mathcal{P}(X)$ be the power set of a set X. We define $L : \mathcal{L}'_\Box \to \mathcal{P}(\mathbb{N})$ to be the function which collects all labels occurring at $\Box$-symbols inside an annotated formula. Precisely, we set recursively:

- $L(p) := L(\bot) := L(\top) := \emptyset$ where $p \in Var$;
- $L(\phi' \land \psi') := L(\phi' \to \psi') := L(\phi' \lor \psi') := L(\phi') \cup L(\psi')$;
- $L(\Box_k\phi') := L(\phi') \cup \{k\}$.

For all annotated formulae ϕ', we now define, recursively, the family of functions $\mathrm{sgn}_{\phi'} : \mathbb{N} \to \mathcal{P}(\{\pm 1\})$. The function $\mathrm{sgn}_{\phi'}(k)$ collects the polarities of $\Box_k$ in ϕ', for every index k. For this, we set for every $k \in \mathbb{N}$:

- $\mathrm{sgn}_p(k) := \mathrm{sgn}_\bot(k) := \mathrm{sgn}_\top(k) := \emptyset$;
- $\mathrm{sgn}_{\phi'\land\psi'}(k) := \mathrm{sgn}_{\phi'\lor\psi'}(k) := \mathrm{sgn}_{\phi'}(k) \cup \mathrm{sgn}_{\psi'}(k)$;
- $\mathrm{sgn}_{\phi'\to\psi'}(k) := \mathrm{sgn}_{\psi'}(k) \cup (-1)\cdot\mathrm{sgn}_{\phi'}(k)$;
- $\mathrm{sgn}_{\Box_l\phi'}(k) := \begin{cases} \mathrm{sgn}_{\phi'}(k) & \text{if } k \neq l\,; \\ \mathrm{sgn}_{\phi'}(k) \cup \{+1\} & \text{else}\,; \end{cases}$

where for $X \in \mathcal{P}(\{\pm 1\})$, we set $(-1)\cdot X := \{(-1)\cdot x \mid x \in X\}$. For negation $\neg$ as a derived connective, this yields:

$$\mathrm{sgn}_{\neg\phi'}(k) = \mathrm{sgn}_{\phi'\to\bot}(k) = \mathrm{sgn}_\bot(k) \cup (-1)\cdot\mathrm{sgn}_{\phi'}(k) = (-1)\cdot\mathrm{sgn}_{\phi'}(k)$$

for every $k \in \mathbb{N}$ as $\mathrm{sgn}_\bot(k) = \emptyset$.

We call an annotated formula ϕ' *properly annotated* (*p.a.*, for short) if every $\Box$-index occurs at most once in it and if $k \in L(\phi')$ is even iff $\mathrm{sgn}_{\phi'}(k) = \{-1\}$. Note that in this case, $\mathrm{sgn}_{\phi'}(k)$ is indeed a singleton for every $k \in L(\phi')$. Note also that being properly annotated is a property inherited by subformulae.

For some formula ϕ and two properly annotated versions ϕ', ϕ'' (that is $(\phi')^\bullet = (\phi'')^\bullet = \phi$), there is a canonical bijection between $L(\phi')$ and $L(\phi'')$ by mapping every index in ϕ' to the *corresponding index* of the same $\Box$-occurrence in ϕ'':

Definition 22. We define the functions $L_{\phi',\phi''} : L(\phi') \to L(\phi'')$ recursively on ϕ as follows:

- for $\phi = \bot$, $\phi = \top$ or $\phi \in Var$: $L_{\phi',\phi''} = \emptyset$;
- for $\phi = \psi \propto \chi$ with $\propto \in \{\wedge, \vee, \to\}$: $L_{\phi',\phi''} = L_{\psi',\psi''} \cup L_{\chi',\chi''}$ where $\phi' = \psi' \propto \chi'$ and $\phi'' = \psi'' \propto \chi''$;
- for $\phi = \Box\psi$: $L_{\phi',\phi''} = L_{\psi',\psi''} \cup (k \mapsto j)$ where $\phi' = \Box_k\psi'$ and $\phi'' = \Box_j\psi''$.

Here we write $(k \mapsto j)$ for the partial function with domain $\{k\}$ and codomain $\{j\}$ which maps k to j.

As ϕ' and ϕ'' are properly annotated, all these functions are well-defined and total on $L(\phi')$ as well as bijective to $L(\phi'')$.

We now formally introduce the concept of a realization of an annotated modal formula.

Definition 23. A *realization* is a partial mapping $r : \mathbb{N} \to Jt_\vartheta$ (or $r : \mathbb{N} \to Jt$). A realization r is called a realization *for* an annotated formula $\phi' \in \mathcal{L}'_\Box$ if all indices of $\Box$'s from ϕ' are assigned under r, that is if $L(\phi') \subseteq \mathrm{dom}(r)$.

A realization r is called *normal* if for every $i \in \mathbb{N}$:

$$\text{if } 2i \in \mathrm{dom}(r), \text{ then } r(2i) = x_i.$$

An annotated formula $\phi' \in \mathcal{L}'_\Box$ has a natural image in $\mathcal{L}_{\vartheta J}$ (or $\mathcal{L}_J$) under some realization r *for* ϕ', written $(\phi')^r$, which we define recursively by the following clauses:

- if $\phi' = p$ or $\phi' = \bot$ or $\phi' = \top$, then $(\phi')^r := \phi$;
- if $\phi' = \psi' \propto \chi'$, then $(\phi')^r := (\psi')^r \propto (\chi')^r$ for $\propto \in \{\wedge, \vee, \to\}$;
- if $\phi' = \Box_k\psi'$, then $(\phi')^r := r(k) : (\psi')^r$.

All these previous concepts naturally generalize to hypersequents in the language $\mathcal{L}'_\Box$, that is *annotated hypersequents*. The maps $\mathcal{I}, \bullet$ naturally carry over by similar definitions in the annotated language. We define $L(G') := L(\mathcal{I}(G'))$, $\mathrm{sgn}_{G'} := \mathrm{sgn}_{\mathcal{I}(G')}$ as well as $(G')^r := (\mathcal{I}(G'))^r$ and call an annotated hypersequent properly annotated if $\mathcal{I}(G')$ is properly annotated. For two proper annotations G', G'' of some hypersequent G, we also introduce the label-bijections $L_{G',G''} := L_{\mathcal{I}(G'),\mathcal{I}(G'')}$ following Definition 22.

8.1. **Substitutions.** In the next subsection, we introduce the previous hypersequent calculi in annotated versions which are then used to constructively provide a realization for a given theorem by induction on the proof. To handle branching rules later on, we use the so-called *merging of realizations*, a technique introduced by Fitting in [14] and applied e.g. by Brünnler, Goetschi and Kuznets in [8] to prove a Realization Theorem for the classical justification logics using a nested sequent calculus.

For this technique, the notion of a substitution of justification variables is necessary. We introduce this notion and present some immediate observations about these substitutions in this subsection.

Definition 24. We call any map $\sigma : V \to Jt_\vartheta$ a *(justification) substitution.* σ naturally extends to two functions mapping terms to terms or formulae to formulae by simultaneously replacing all occurring justification variables by their respective images under σ. We identify these extensions with σ and thus write σ for any such extension as well.

We also write $t\sigma$ or $\phi\sigma$ for the images of terms t or formulae ϕ under σ (or more precisely, under the extensions identified with σ), respectively.

Given a substitution σ, we write $\mathrm{dom}(\sigma) := \{x \in V \mid \sigma(x) \neq x\}$. Given two substitutions σ, σ', we write $\sigma\sigma'$ for the map

$$x \mapsto \sigma(x)\sigma' = \sigma'(\sigma(x)).$$

Given a proof calculus for a justification logic, it is natural to require that being a theorem is invariant under substitutions, especially in the context of the Realization Theorem. For this, let

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0\}$$

and $\mathcal{S}_0$ be either $\mathcal{GJL}_0$ or $\mathcal{PGJL}_0$. Now, if $\mathcal{S}_{CS} \vdash \phi$ for some axiomatically appropriate constant specification CS for $\mathcal{S}_0$, then for some substitution σ we in general only have $\mathcal{S}_{CS\sigma} \vdash \phi\sigma$ where $CS\sigma := \{c : \phi\sigma \mid c : \phi \in CS\}$ (see [25] regarding this statement in the classical case). So, for closure under substitutions, we need the constant specification to be closed under substitutions itself. One way to enforce this is by requiring that the constant specification in question is schematic (see Section 2):

Lemma 46 (Substitution Lemma)**.** *Let CS be a schematic constant specification for $\mathcal{S}_0$. Let $\phi \in \mathcal{L}_{\vartheta J}$ ($\phi \in \mathcal{L}_J$) such that $\vdash_{\mathcal{S}_{CS}} \phi$. Then, for any substitution σ: $\vdash_{\mathcal{S}_{CS}} \phi\sigma$.*

The proof is a straightforward induction on the length of the proof and can be obtained through an easy generalization of the proof for the classical case (see [14]).

The following remark collects some important facts on substitutions and realizations which we will need for handling annotated rules later on.

Remark 8. Let σ, σ' be substitutions, r be a realization and $\phi' \in \mathcal{L}'_\Box$ be p.a. such that r is a realization *for* ϕ'.

(1) $\sigma \circ r$ is a realization (for ϕ')[8];
(2) $\sigma \circ r$ is normal iff $\forall k \in \mathbb{N} : x_k \in \mathrm{dom}(\sigma)$ implies $2k \notin \mathrm{dom}(r)$;
(3) $(\phi')^r\sigma = (\phi')^{\sigma \circ r}$;
(4) if
    (i) $\sigma|_D = \sigma'|_D$ for $D := \mathrm{dom}(\sigma) \cap \mathrm{dom}(\sigma')$, and
    (ii) $\mathrm{jvar}(\sigma(x)) \cup \mathrm{jvar}(\sigma'(x)) \subseteq \{x\}$ for all $x \in V$,
    then $\sigma\sigma' = \sigma'\sigma$.

To see item (4), note that $\sigma(\sigma'(x)) = \sigma'(\sigma(x))$ is trivial for $x \notin \mathrm{dom}(\sigma) \cup \mathrm{dom}(\sigma')$.

If $x \in \mathrm{dom}(\sigma) \setminus \mathrm{dom}(\sigma')$, then we have

$$\sigma'(\sigma(x)) = \sigma(x) = \sigma(\sigma'(x))$$

by (ii), as $\mathrm{jvar}(\sigma(x)) \subseteq \{x\}$ and as $\sigma'(x) = x$ by assumption. A similar argument can be used to show $\sigma(\sigma'(x)) = \sigma'(\sigma(x))$ for $x \in \mathrm{dom}(\sigma) \setminus \mathrm{dom}(\sigma')$.

Lastly, if $x \in D$, then $\sigma(x) = \sigma'(x)$ by (i). Thus

$$\sigma'(\sigma(x)) = \sigma'(\sigma'(x)) = \sigma(\sigma'(x))$$

where the second equality follows again through $\sigma(x) = \sigma'(x)$ from (i) and through $\mathrm{jvar}(\sigma(x)) \subseteq \{x\}$ from (ii).

We can now state and prove the theorem on merging of realizations. This result gives us the possibility to combine two realizations into one in various ways, depending on the sign of subformulae in some underlying context formula. For this, we restrict ourselves to schematic and axiomatically appropriate constant specification to be able to use substitutions and the Internalization Property. The requirement of being axiomatically appropriate is no real drawback while aiming for a Realization Theorem either, as by the modal inference rule ($N\Box$), in all of the weak or standard Gödel modal logics, $\Box\theta$ is a theorem for every theorem θ. Thus, any candidate Gödel justification logic for realizing a corresponding (weak or standard) Gödel modal logic has to have the Internalization Property.

Theorem 47 (Realization Merging Theorem)**.** *Let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Let $\phi' \in \mathcal{L}'_\Box$ be p.a. and r_1, r_2 be normal realizations for ϕ'. Then, there is a normal realization r for ϕ' and a substitution σ such that:*

(1) $\forall x \in V : \mathrm{jvar}(\sigma(x)) \subseteq \{x\}$*;*
(2) $\mathrm{dom}(\sigma) \subseteq \{x_n \mid 2n \in L(\phi')\}$*;*
(3) for each subformula ψ' of ϕ':
    (a) if ψ' is a positive subformula of ϕ', then $\vdash_{\mathcal{S}_{CS}} (\psi')^{r_i}\sigma \to (\psi')^r$,
    (b) if ψ' is a negative subformula of ϕ', then $\vdash_{\mathcal{S}_{CS}} (\psi')^r \to (\psi')^{r_i}\sigma$,
    where $i = 1, 2$.

Proof. At first, we assume that r_1 and r_2 are (using Fitting's terminology from [14]) *non-self-referential on variables over* ϕ', that is we require that

$$\Box_{2n}\psi' \in \mathrm{sf}(\phi') \text{ implies } x_n \notin \mathrm{jvar}((\psi')^{r_j})$$

for $j = 1, 2$. Assuming this, we construct realization/substitution pairs $(r_{\psi'}, \sigma_{\psi'})$ for the subformulae $\psi' \in \mathrm{sf}(\phi')$ by recursion on the structure of ψ' such that $r_{\psi'}$ is normal and:

(1) $\forall x \in V : \mathrm{jvar}(\sigma_{\psi'}(x)) \subseteq \{x\}$;
(2) $\mathrm{dom}(\sigma_{\psi'}) \subseteq \{x_n \mid 2n \in L(\psi')\}$;
(3) for all $\chi' \in \mathrm{sf}(\psi')$:
    (a) if χ' is a positive subformula of ϕ', then $\vdash_{\mathcal{S}_{CS}} (\chi')^{r_i}\sigma_{\psi'} \to (\chi')^{r_{\psi'}}$ for $i = 1, 2$;
    (b) if χ' is a negative subformula of ϕ', then $\vdash_{\mathcal{S}_{CS}} (\chi')^{r_{\psi'}} \to (\chi')^{r_i}\sigma_{\psi'}$ for $i = 1, 2$.

[8]Here, and in the following, we write $\circ$ for the composition of functions. Note that as before, σ is representative for its extension to a function from terms to terms.

For the recursion base, we may set $r_{\psi'} := \emptyset$ and $\sigma_{\psi'} := \mathrm{id}_V$ for ψ' being an atomic subformula of ϕ'. Clearly these $(r_{\psi'}, \sigma_{\psi'})$ satisfy the conditions (1)-(3) and $r_{\psi'}$ is normal.

Now, suppose that α', β' are subformulae of ϕ' and that together with realization/substitution pairs $(r_{\alpha'}, \sigma_{\alpha'}), (r_{\beta'}, \sigma_{\beta'})$ they satisfy (1)-(3) and that $r_{\alpha'}$, $r_{\beta'}$ are normal. Then, we have the following case-distinctions:

- If $\psi' = \alpha' \wedge \beta'$ is a subformula of ϕ', we define

$$r_{\psi'}(n) := \begin{cases} r_{\alpha'}(n)\sigma_{\beta'} & \text{if } n \in L(\alpha'), \\ r_{\beta'}(n)\sigma_{\alpha'} & \text{if } n \in L(\beta'), \end{cases}$$

  and $\sigma_{\psi'} := \sigma_{\alpha'}\sigma_{\beta'}$. Note first, that as ϕ' is properly annotated and as $\psi' = \alpha' \wedge \beta'$ is a subformula of ϕ', we have $L(\alpha') \cap L(\beta') = \emptyset$. Thus, as we have

$$\mathrm{dom}(\sigma_{\alpha'}) \subseteq \{x_n \mid 2n \in L(\alpha')\} \text{ and } \mathrm{dom}(\sigma_{\beta'}) \subseteq \{x_n \mid 2n \in L(\beta')\}$$

  by induction hypothesis, we obtain $\mathrm{dom}(\sigma_{\alpha'}) \cap \mathrm{dom}(\sigma_{\beta'}) = \emptyset$ and by item (4) of Remark 8, we get $\sigma_{\psi'} = \sigma_{\alpha'}\sigma_{\beta'} = \sigma_{\beta'}\sigma_{\alpha'}$. Naturally, the conditions (1) and (2) are satisfied for $\sigma_{\psi'}$. Also, by the above, it is straightforward to see that $r_{\psi'}$ is normal: $r_{\alpha'}, r_{\beta'}$ are normal by induction hypothesis. Let $2n \in \mathrm{dom}(r_{\psi'})$, i.e. $2n \in L(\alpha') \cup L(\beta')$. If $2n \in L(\alpha')$, then $r_{\alpha'}(2n) = x_n$ as $r_{\alpha'}$ is normal. As $L(\alpha') \cap L(\beta') = \emptyset$, we have $2n \notin L(\beta')$. By condition (2) applied to $\sigma_{\beta'}$, we have $\sigma_{\beta'}(x_n) = x_n$. Therefore, we have $r_{\psi'}(2n) = r_{\alpha'}(2n)\sigma_{\beta'} = x_n$. We similarly get this for $2n \in L(\beta')$. Therefore, $r_{\psi'}$ is normal.

  For (3), let $\chi' \in \mathrm{sf}(\psi')$, i.e. $\chi' \in \mathrm{sf}(\alpha') \cup \mathrm{sf}(\beta')$ or $\chi' = \psi'$. Suppose the former and that $\chi' \in \mathrm{sf}(\alpha')$. The case of $\chi' \in \mathrm{sf}(\beta')$ follows analogously. We divide again between the following two cases:

  – If χ' is negative in ϕ', then

$$\vdash_{\mathcal{S}_{CS}} (\chi')^{r_{\alpha'}} \to (\chi')^{r_j}\sigma_{\alpha'}, \quad j = 1, 2.$$

  By the Substitution Lemma, we have

$$\vdash_{\mathcal{S}_{CS}} (\chi')^{r_{\alpha'}}\sigma_{\beta'} \to (\chi')^{r_j}\sigma_{\alpha'}\sigma_{\beta'}, \quad j = 1, 2.$$

  Per definition of $r_{\psi'}$ and $\sigma_{\psi'}$, this amounts exactly to

$$\vdash_{\mathcal{S}_{CS}} (\chi')^{r_{\psi'}} \to (\chi')^{r_j}\sigma_{\psi'}, \quad j = 1, 2.$$

  – If χ' is positive in ϕ', then similarly as before

$$\vdash_{\mathcal{S}_{CS}} (\chi')^{r_j}\sigma_{\alpha'} \to (\chi')^{r_{\alpha'}}, \quad j = 1, 2,$$

  and therefore by using the substitution $\sigma_{\beta'}$ again, we also obtain

$$\vdash_{\mathcal{S}_{CS}} (\chi')^{r_j}\sigma_{\psi'} \to (\chi')^{r_{\psi'}}$$

  for $j = 1, 2$.

  Now suppose $\chi' = \psi'$. Then, we again divide between ψ' being positive or negative in ϕ'.

  – Suppose ψ' is negative in ϕ'. Then, α' and β' are negative in ϕ' and thus we have, as above, that

$$\vdash_{\mathcal{S}_{CS}} (\alpha')^{r_{\psi'}} \to (\alpha')^{r_j}\sigma_{\psi'} \text{ and } \vdash_{\mathcal{S}_{CS}} (\beta')^{r_{\psi'}} \to (\beta')^{r_j}\sigma_{\psi'}$$

  for $j = 1, 2$. Therefore, we get

$$\vdash_{\mathcal{S}_{CS}} \left((\alpha')^{r_{\psi'}} \wedge (\beta')^{r_{\psi'}}\right) \to \left((\alpha')^{r_j}\sigma_{\psi'} \wedge (\beta')^{r_j}\sigma_{\psi'}\right) \quad (j = 1, 2)$$

  and hence

$$\vdash_{\mathcal{S}_{CS}} (\alpha' \wedge \beta')^{r_{\psi'}} \to (\alpha' \wedge \beta')^{r_j}\sigma_{\psi'}$$

  for $j = 1, 2$, which is the claim.

  – If ψ' is positive in ϕ'. We similarly have

$$\vdash_{\mathcal{S}_{CS}} (\alpha')^{r_j}\sigma_{\psi'} \to (\alpha')^{r_{\psi'}} \text{ and } \vdash_{\mathcal{S}_{CS}} (\beta')^{r_j}\sigma_{\psi'} \to (\beta')^{r_{\psi'}}$$

  for $j = 1, 2$ and therefore

$$\vdash_{\mathcal{S}_{CS}} \left((\alpha')^{r_j}\sigma_{\psi'} \wedge (\beta')^{r_j}\sigma_{\psi'}\right) \to \left((\alpha')^{r_{\psi'}} \wedge (\beta')^{r_{\psi'}}\right)$$

  for $j = 1, 2$.

- If $\psi' = \alpha' \vee \beta'$ is a subformula of ϕ' we again define

$$r_{\psi'}(n) := \begin{cases} r_{\alpha'}(n)\sigma_{\beta'} & \text{if } n \in L(\alpha'), \\ r_{\beta'}(n)\sigma_{\alpha'} & \text{if } n \in L(\beta'), \end{cases}$$

  and $\sigma_{\psi'} := \sigma_{\alpha'}\sigma_{\beta'}$ as in the case of $\alpha' \wedge \beta'$. Note that again $\sigma_{\alpha'}\sigma_{\beta'} = \sigma_{\beta'}\sigma_{\alpha'}$. Then, the properties (1)-(3) and normality of $r_{\psi'}$ follow as in the previous case for $\alpha' \wedge \beta'$. In particular, the case of $\chi' = \psi'$

for item (3) can be obtained by following the same line of reasoning from the $\wedge$-case, while using the following theorem of $\mathcal{G}$ (and thus of $\mathcal{S}_{CS}$):

$$\vdash_{\mathcal{S}_{CS}} ((\theta_1 \to \theta_2) \wedge (\theta_3 \to \theta_4)) \to ((\theta_1 \vee \theta_3) \to (\theta_2 \vee \theta_4)).$$

- If $\psi' = \alpha' \to \beta'$ is a subformula of ϕ', then we again set $\sigma_{\psi'} := \sigma_{\alpha'}\sigma_{\beta'}$ (also here commutativity applies) as well as
$$r_{\psi'}(n) := \begin{cases} r_{\alpha'}(n)\sigma_{\beta'} & \text{if } n \in L(\alpha'),\\ r_{\beta'}(n)\sigma_{\alpha'} & \text{if } n \in L(\beta'). \end{cases}$$
The properties (1) and (2) for $\sigma_{\psi'}$ follow as in the two previous cases. This $r_{\psi'}$ can be shown to be normal by similar reasoning as in the $\psi' = \alpha' \wedge \beta'$-case. So we only focus on the instance $\chi' = \psi'$ of item (3). This case differs (a little) from the $\wedge$(or $\vee$)-case and we thus give some details.

  If ψ' is negative in ϕ', then α' is positive and β' is negative in ϕ'. Thus, we have by construction of $(r_{\psi'}, \sigma_{\psi'})$ that
$$\vdash_{\mathcal{S}_{CS}} (\alpha')^{r_j}\sigma_{\psi'} \to (\alpha')^{r_{\psi'}} \text{ and } \vdash_{\mathcal{S}_{CS}} (\beta')^{r_{\psi'}} \to (\beta')^{r_j}\sigma_{\psi'}$$
for $j = 1, 2$ by similar reasoning as in the $\wedge$(or $\vee$)-case. By propositional reasoning this yields
$$\vdash_{\mathcal{S}_{CS}} \left((\alpha')^{r_{\psi'}} \to (\beta')^{r_{\psi'}}\right) \to \left((\alpha')^{r_j}\sigma_{\psi'} \to (\beta')^{r_j}\sigma_{\psi'}\right) \quad (j = 1, 2)$$
which is exactly $\vdash_{\mathcal{S}_{CS}} (\psi')^{r_{\psi'}} \to (\psi')^{r_j}\sigma_{\psi'}$ for $j = 1, 2$.

  If otherwise ψ' is positive in ϕ', then α' is negative and β' is positive in ϕ' and the argument is similar to the above with the roles of α' and β' reversed.
- If $\psi' = \Box_i\alpha'$, we now already divide in the construction over whether ψ' is a positive or a negative subformula of ϕ':
  - if ψ' is a positive subformula, then α' is also a positive subformula and thus by assumption
  $$\vdash_{\mathcal{S}_{CS}} (\alpha')^{r_j}\sigma_{\alpha'} \to (\alpha')^{r_{\alpha'}}$$
  for $j = 1, 2$ where then, by Internalization in $\mathcal{S}_{CS}$, there are *closed* justification terms t_1, t_2 such that
  $$(\dagger) \qquad \vdash_{\mathcal{S}_{CS}} t_j : \left((\alpha')^{r_j}\sigma_{\alpha'} \to (\alpha')^{r_{\alpha'}}\right)$$
  for $j = 1, 2$. We remind on Remark 2 (which also applies to the positive versions) that the terms t_1, t_2 indeed *can* be chosen to be closed. We set $\sigma_{\psi'} := \sigma_{\alpha'}$ and
  $$r_{\psi'}(n) := \begin{cases} [[t_1 \cdot r_1(i)\sigma_{\alpha'}] + [t_2 \cdot r_2(i)\sigma_{\alpha'}]] & \text{if } n = i,\\ r_{\alpha'}(n) & \text{if } n \in L(\alpha'). \end{cases}$$
  This $r_{\psi'}$ is normal: $r_{\alpha'}$ is normal by induction hypothesis and i is odd since ψ' is positive in ϕ' and ϕ' is properly annotated, i.e. in particular $\mathrm{sgn}_{\phi'}(i) = \{+1\}$. Items (1) and (2) follow directly from the definition of $\sigma_{\psi'}$ by the induction hypothesis applied to α' and $\sigma_{\alpha'}$. We again focus on (3) where we only consider the case of $\chi' = \psi'$ as before. The other cases are immediate as $r_{\psi'}$ respects $r_{\alpha'}$ on $L(\alpha')$. Using (†), we obtain
  $$\vdash_{\mathcal{S}_{CS}} r_j(i)\sigma_{\alpha'} : (\alpha')^{r_j}\sigma_{\alpha'} \to [t_j \cdot r_j(i)\sigma_{\alpha'}] : (\alpha')^{r_{\alpha'}}$$
  for $j = 1, 2$ and thus
  $$\vdash_{\mathcal{S}_{CS}} r_j(i)\sigma_{\alpha'} : (\alpha')^{r_j}\sigma_{\alpha'} \to [[t_1 \cdot r_1(i)\sigma_{\alpha'}] + [t_2 \cdot r_2(i)\sigma_{\alpha'}]] : (\alpha')^{r_{\alpha'}}$$
  by the axiom schemes (J) and $(+)$ in $\mathcal{S}_{CS}$ where the latter is exactly $\vdash_{\mathcal{S}_{CS}} (\Box_i\alpha')^{r_j}\sigma_{\alpha'} \to (\Box_i\alpha')^{r_{\psi'}}$ for $j = 1, 2$.
  - if ψ' is a negative subformula, then α' is also a negative subformula of ϕ' and thus by assumption
  $$\vdash_{\mathcal{S}_{CS}} (\alpha')^{r_{\alpha'}} \to (\alpha')^{r_j}\sigma_{\alpha'}$$
  for $j = 1, 2$ where again, by the Internalization Property, we deduce that there are *closed* terms s_1, s_2 such that
  $$(\ddagger) \qquad \vdash_{\mathcal{S}_{CS}} s_j : \left((\alpha')^{r_{\alpha'}} \to (\alpha')^{r_j}\sigma_{\alpha'}\right)$$
  for $j = 1, 2$. We take
  $$r_{\psi'}(n) := \begin{cases} x_k & \text{if } n = i,\\ r_{\alpha'}(n) & \text{if } n \in L(\alpha'), \end{cases}$$
  and
  $$\sigma_{\psi'}(x_n) := \begin{cases} [[s_1 + s_2] \cdot x_k] & \text{if } n = k,\\ \sigma_{\alpha'}(x_n) & \text{otherwise,} \end{cases}$$

where $k := i/2$. Note that i has to be even as $\mathrm{sgn}_{\phi'}(i) = \{-1\}$, following the assumption that ψ' is negative in ϕ'. Naturally, $r_{\psi'}$ is normal as $r_{\alpha'}$ is by induction hypothesis and as $r_{\psi'}(2k) = x_k$. For item (1), note that the terms s_1, s_2 are assumed to be closed.
This $\sigma_{\psi'}$ also naturally satisfies item (2): if $x \in \mathrm{dom}(\sigma_{\psi'})$, then either $x \in \mathrm{dom}(\sigma_{\alpha'})$ or $x = x_k$. For the former, (2) follows from the induction hypothesis applied to α'. For the latter, clearly $2k = i \in L(\psi')$ and thus $x_k \in \{x_n \mid 2n \in L(\psi')\}$.
We verify item (3) for $\chi' = \psi'$. Again, as i is even, we have $r_j(i) = x_k$. By (‡), we then derive the following for $j = 1, 2$ using the axiom scheme (+):

$$\vdash_{\mathcal{S}_{CS}} [s_1 + s_2] : \left((\alpha')^{r_{\alpha'}} \to (\alpha')^{r_j}\sigma_{\alpha'}\right).$$

Using this and the axiom scheme (J), we then obtain

$$\vdash_{\mathcal{S}_{CS}} x_k : (\alpha')^{r_{\alpha'}} \to [[s_1 + s_2] \cdot x_k] : (\alpha')^{r_j}\sigma_{\alpha'}$$

for $j = 1, 2$. Note that it holds that $[[s_1 + s_2] \cdot x_k] : (\alpha')^{r_j}\sigma_{\alpha'} = \left(x_k : (\alpha')^{r_j}\right)\sigma_{\psi'}$ as we have $(\alpha')^{r_j}\sigma_{\alpha'} = (\alpha')^{r_j}\sigma_{\psi'}$. To see the latter, suppose $(\alpha')^{r_j}\sigma_{\alpha'} \neq (\alpha')^{r_j}\sigma_{\psi'}$. Then, as $\sigma_{\psi'}$ and $\sigma_{\alpha'}$ are equal on everything but x_k, we have to have $x_k \in \mathrm{jvar}((\alpha')^{r_j})$. But as $\Box_{2k}\alpha' = \Box_i\alpha' = \psi' \in \mathrm{sf}(\phi')$, this is impossible as r_1, r_2 are assumed to be non-self-referential on variables over ϕ'. Therefore the last derivation translates to

$$\vdash_{\mathcal{S}_{CS}} (\Box_i\alpha')^{r_{\psi'}} \to (\Box_i\alpha')^{r_j}\sigma_{\psi'}$$

for $j = 1, 2$.

If r_1 and r_2 were non-self-referential on variables over ϕ', then the desired realization/substitution pair is given by $(r_{\phi'}, \sigma_{\phi'}) = (r, \sigma)$.

If r_1 and r_2 failed to be non-self-referential on variables over ϕ', then we construct two new realizations $\hat{r}_1$, $\hat{r}_2$ as follows: let $j = 1, 2$ and define

$$P_j := \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} : \Box_{2k}\psi' \in \mathrm{sf}(\phi'), n \in L(\psi'), x_k \in \mathrm{jvar}(r_j(n))\}.$$

These $n \in P_j$ are the indices where non-self-referentiality fails in r_j. We will temporarily replace those by *new* justification variables and for this, we define

$$V_{P_j} := \left\{x_{p_1^{(j)}}, \ldots, x_{p_{|P_j|}^{(j)}}\right\} \subseteq V$$

with $p_i^{(j)} < p_{i+1}^{(j)}$ and where

$$(V_{P_1} \cup V_{P_2}) \cap (\mathrm{jvar}((\phi')^{r_1}) \cup \mathrm{jvar}((\phi')^{r_2})) = \emptyset \text{ and } V_{P_1} \cap V_{P_2} = \emptyset.$$

Essentially, this last line just says that all these variables contained in $V_{P_1} \cup V_{P_2}$ are distinct and not occurring in $(\phi')^{r_j}$ for $j = 1, 2$ (which is what we meant with *new* in the above). But this precise notation makes the following definitions more succinct.

Now, for $P_j = \left\{n_1^{(j)}, \ldots, n_{|P_j|}^{(j)}\right\}$ with $n_i^{(j)} < n_{i+1}^{(j)}$, we define

$$\hat{r}_j\left(n_i^{(j)}\right) = x_{p_i^{(j)}}$$

for $i \in \{1, \ldots, |P_j|\}$ and $\hat{r}_j(n) = r_j(n)$ for $n \notin P_j$. Note that these $\hat{r}_j$ are both normal as r_j is normal and every $n \in P_j$ has to be odd: if n is even, say $n = 2m$, then $r_j(n) = x_m$ and if $\Box_{2k}\psi' \in \mathrm{sf}(\phi')$ with $n \in L(\psi')$, then $k \neq m$ as ϕ' is properly annotated. Therefore $x_k \notin \mathrm{jvar}(r_j(n))$.

These $\hat{r}_j$ are now non-self-referential on variables over ϕ' by construction. Using these temporarily modified versions of the original r_j, we construct a realization/substitution pair $(\hat{r}_{\phi'}, \hat{\sigma}_{\phi'})$ for ϕ' with respect to $\hat{r}_1, \hat{r}_2$ by the recursive process from the first part of the proof.

Using this resulting $(\hat{r}_{\phi'}, \hat{\sigma}_{\phi'})$, we then define two substitutions σ', σ'' via

$$\sigma'\left(x_{p_i^{(j)}}\right) = r_j\left(n_i^{(j)}\right)\hat{\sigma}_{\phi'} \text{ and } \sigma''\left(x_{p_i^{(j)}}\right) = r_j\left(n_i^{(j)}\right)$$

for $x_{p_i^{(j)}} \in V_{P_1} \cup V_{P_2}$ and via $\sigma'(x) = \sigma''(x) = x$ for $x \notin V_{P_1} \cup V_{P_2}$. Both of these substitutions are well-defined as $V_{P_1} \cap V_{P_2} = \emptyset$. We define $r := \sigma' \circ \hat{r}_{\phi'}$ as well as $\sigma := \hat{\sigma}_{\phi'}$ and the desired realization substitution pair is now given by (r, σ). σ' serves to undo the previous introduction of new variables. To see that (r, σ) is the desired realization/substitution pair corresponding to the original r_1, r_2, we verify the respective properties (1) - (3) for (r, σ) over ϕ' with respect to r_1, r_2.

By construction of $(\hat{r}_{\phi'}, \hat{\sigma}_{\phi'})$, $\sigma = \hat{\sigma}_{\phi'}$ satisfies (1) and (2). For (3), we first show two properties of σ' and σ'':

(i) $\sigma''\hat{\sigma}_{\phi'} = \hat{\sigma}_{\phi'}\sigma'$;

(ii) $\sigma'' \circ \hat{r}_j = r_j$.

Item (ii) is quite immediate: as all variables from $V_{P_1} \cup V_{P_2}$ are new, we obtain, since $\mathrm{dom}(\sigma'') \subseteq V_{P_1} \cup V_{P_2}$, that $\hat{r}_j(n)\sigma'' = r_j(n)\sigma'' = r_j(n)$ for any $n \notin P_1 \cup P_2$ by definition of $\hat{r}_j$. For $n \in P_j$ for $j = 1$ or $j = 2$, say $n = n_i^{(j)}$, we have

$$\hat{r}_j\left(n_i^{(j)}\right)\sigma'' = \sigma''\left(x_{p_i^{(j)}}\right) = r_j\left(n_i^{(j)}\right)$$

by definition.

For item (i), note that, by definition, we have

$$\left(x_{p_i^{(j)}}\right)\sigma''\hat{\sigma}_{\phi'} = r_j\left(n_i^{(j)}\right)\hat{\sigma}_{\phi'} = \left(x_{p_i^{(j)}}\right)\sigma' = \left(x_{p_i^{(j)}}\right)\hat{\sigma}_{\phi'}\sigma'.$$

Here, everything but the last equality follows directly from the definitions. For this last equality, note that $\hat{\sigma}_{\phi'}$ fulfills item (2) of the Realization Merging Theorem by construction, that is we have

$$\mathrm{dom}(\hat{\sigma}_{\phi'}) \subseteq \{x_n \mid 2n \in L(\phi')\}.$$

So, as $x_{p_i^{(j)}}$ is a new variable, we in particular have $\hat{\sigma}_{\phi'}\left(x_{p_i^{(j)}}\right) = x_{p_i^{(j)}}$ which gives the last equality.

Now, we want to establish item (3) for the pair (r, σ) with respect to the original r_1, r_2. We at first have

$$\vdash_{\mathcal{S}_{CS}} (\psi')^{\hat{r}_j}\sigma \to (\psi')^{\hat{r}_{\phi'}}$$

from (3) applied to $(\hat{r}_{\phi'}, \hat{\sigma}_{\phi'})$ and $\hat{r}_1, \hat{r}_2$ (as $\sigma = \hat{\sigma}_{\phi'}$) for ψ' being a positive subformula of ϕ'. We apply the substitution σ' to obtain

$$\vdash_{\mathcal{S}_{CS}} (\psi')^{\hat{r}_j}\sigma\sigma' \to (\psi')^{\hat{r}_{\phi'}}\sigma'$$

By item (i) and (ii), we have $(\psi')^{\hat{r}_j}\sigma\sigma' = (\psi')^{\hat{r}_j}\sigma''\sigma = (\psi')^{r_j}\sigma$ and therefore, as $(\psi')^{\hat{r}_{\phi'}}\sigma' = (\psi')^{r_{\phi'}}$ by definition, we obtain

$$\vdash_{\mathcal{S}_{CS}} (\psi')^{r_j}\sigma \to (\psi')^{r_{\phi'}}.$$

The argument is similar if ψ' is a negative subformula of ϕ'. □

The above theorem was obtained, for the classical justification logic $\mathcal{LP}$, by Fitting in [14]. Brünnler, Goetschi and Kuznets state it without proof in [8] for the other common classical justification logics. The proof presented here is a simplified version of that of Fitting from [14], adapted to the case of the Gödel justification logics. However, one may note that the pre-linearity scheme is not needed in the proof and it thus stays valid in the context of the intuitionistic justification logics as defined in [26].

We decided to follow this approach of Fitting towards realization as for one, the Realization Merging Theorem can be used to provide further constructive insights into the justification logics in questions (see [14]) and, for another, the proof of the Realization Theorem via the Realization Merging Theorem is modular in a way which nicely accommodates the multitude of systems considered by us.

In the setting of the Realization Merging Theorem, we also say that the realization/substitution pair (r, σ) *hereditarily merges r_1 and r_2 on ϕ'*.

8.2. **Annotated calculi.** In the following, let

$$\mathcal{HGML}^{\pm}_{\Box} \in \{\mathcal{HGK}_{\Box}, \mathcal{HGT}_{\Box}, \mathcal{HGK}4_{\Box}, \mathcal{HGS}4_{\Box}, \mathcal{HGK}^{-}_{\Box}, \mathcal{HGT}^{-}_{\Box}, \mathcal{HGK}4^{-}_{\Box}, \mathcal{HGS}4^{-}_{\Box}\}.$$

We may define $\mathcal{HGML}^{\pm}_{\Box}$ in an annotated version over the language $\mathcal{L}'_{\Box}$, denoted by $\mathcal{HGML}^{\pm\prime}_{\Box}$, which results from the same rules as $\mathcal{HGML}^{\pm}_{\Box}$ (in the language $\mathcal{L}'_{\Box}$), but with the rules $(\Box)$, $(\Box^-)$, $(\Box\rhd)$, $(\rhd\Box^-)_1$, $(\rhd\Box^-)_2$,

$(\rhd\Box)_1$ and $(\rhd\Box)_2$ respectively replaced by corresponding annotated versions:

$$\frac{\pi_1',\ldots,\pi_m'\rhd\mid\gamma_1',\ldots,\gamma_n'\rhd\phi'}{\Box_{j_1}\pi_1',\ldots,\Box_{j_m}\pi_m'\rhd\mid\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\Box_k\phi'}\ (\Box)';$$

$$\frac{\gamma_1',\ldots,\gamma_n'\rhd\phi'}{\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\Box_k\phi'}\ (\Box^-)';$$

$$\frac{G'\mid\phi',\Gamma'\rhd\Delta'}{G'\mid\Box_k\phi',\Gamma'\rhd\Delta'}\ (\Box\rhd)';$$

$$\frac{\gamma_1',\ldots,\gamma_n',\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\phi'}{\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\Box_k\phi'}\ (\rhd\Box^-)_1';$$

$$\frac{\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\phi'}{\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\Box_k\phi'}\ (\rhd\Box^-)_2';$$

$$\frac{\pi_1',\ldots,\pi_m',\Box_{j_1}\pi_1',\ldots,\Box_{j_m}\pi_m'\rhd\mid\gamma_1',\ldots,\gamma_n',\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\phi'}{\Box_{j_1}\pi_1',\ldots,\Box_{j_m}\pi_m'\rhd\mid\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\Box_k\phi'}\ (\rhd\Box)_1';$$

$$\frac{\Box_{j_1}\pi_1',\ldots,\Box_{j_m}\pi_m'\rhd\mid\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\phi'}{\Box_{j_1}\pi_1',\ldots,\Box_{j_m}\pi_m'\rhd\mid\Box_{i_1}\gamma_1',\ldots,\Box_{i_n}\gamma_n'\rhd\Box_k\phi'}\ (\rhd\Box)_2'.$$

We present these annotated modal rules by expanding the internal multisets into their constituting formulae. We do this, as we want (and need) the freedom to index every formula in a multiset, counting multiplicities, by a different annotated instance of $\Box$ (as in Fitting's [14]).

There is a natural notion of proof over these annotated calculi, this time defined via *annotated* trees of annotated hypersequents. We use the same notation as in the context of the unannotated calculi.

We obtain the following lemma via an induction over the length of the proof:

Lemma 48. *Let G' be an annotated hypersequent. If $\gamma'\vdash_{\mathcal{HGML}_\Box^{\pm\prime}} G'$, then $\gamma'^\bullet\vdash_{\mathcal{HGML}_\Box^{\pm}} G'^\bullet$.*

Of particular interest however is the reversal of the above statement.

Lemma 49. *If $\gamma\vdash_{\mathcal{HGML}_\Box^{\pm}} G$, then for any annotation G' of G, there is an annotated proof γ' such that $\gamma'\vdash_{\mathcal{HGML}_\Box^{\pm\prime}} G'$ and $\gamma'^\bullet=\gamma$.*

Proof. We fix an annotation G' of G and propagate upwards through the proof tree, annotating the intermediate sequents in the proof. The annotation of the premise hypersequents are always uniquely determined by the annotation of the conclusion, as we do not consider the cut-rule. □

Note that the condition $\gamma'^\bullet=\gamma$ especially implies that the proofs have the same length. This will be of importance in the proof of the Realization Theorem later on.

9. The realization results

In this section, we prove the Realization Theorem by recursively constructing realizations over proofs in the annotated calculi. For this, the following lemmas first provide results on how the Gödel justification logics realize inference in the style of the various hypersequent rules.

Thus, in this section (if not stated otherwise) let

$$S_0\in\{\mathcal{GJ}_0,\mathcal{GJT}_0,\mathcal{GJ}4_0,\mathcal{GLP}_0,\mathcal{PGJ}_0,\mathcal{PGJT}_0,\mathcal{PGJ}4_0,\mathcal{PGLP}_0\}$$

and CS be a schematic and axiomatically appropriate and constant specification for $\mathcal{S}_0$. We only consider the more complicated rules, i.e. those where either merging or modal reasoning is required in constructing the realization.

9.1. **Structural rules.** Regarding the structural rules, we only consider the (COM)-rule. Not only does it require the Realization Merging Theorem but the rule is actually a hypersequent counterpart to the pre-linearity scheme $(\phi\to\psi)\vee(\psi\to\phi)$ in propositional Gödel logic. This scheme is of greater importance as it actually is the only axiom needed in addition to some formulation of the intuitionistic propositional calculus to axiomatize $[0,1]$-valued propositional Gödel logics.

Besides the approach over t-norm based semantics and Hájek's basic logic $\mathcal{BL}$, this is the other main semantical access point to Gödel many-valued logics.

Lemma 50 ((COM)-case)**.** *Let $H':=G'\mid\Gamma_1',\Gamma_2'\rhd\Delta_1'\mid\Pi_1',\Pi_2'\rhd\Delta_2'$ be a p.a. hypersequent and suppose there are normal realizations r and s for $G'\mid\Gamma_1',\Pi_1'\rhd\Delta_1'$ and $G'\mid\Gamma_2',\Pi_2'\rhd\Delta_2'$, respectively, such that*

$$\vdash_{\mathcal{S}_{CS}}(G'\mid\Gamma_1',\Pi_1'\rhd\Delta_1')^r\ \textit{and}\ \vdash_{\mathcal{S}_{CS}}(G'\mid\Gamma_2',\Pi_2'\rhd\Delta_2')^s.$$

Then, there is a normal realization t for H' such that $\vdash_{\mathcal{S}_{CS}}(H')^t$.

Proof. By the Realization Merging Theorem, there exists a normal realization u for G' together with a substitution σ such that u and σ hereditarily merge r and s on G'. In particular, we have

$$\vdash_{\mathcal{S}_{CS}} (G')^r\sigma \to (G')^u \text{ and } \vdash_{\mathcal{S}_{CS}} (G')^s\sigma \to (G')^u.$$

We set

$$t(k) := \begin{cases} r(k)\sigma & \text{if } k \in L(\Gamma_1', \Pi_1' \rhd \Delta_1'), \\ s(k)\sigma & \text{if } k \in L(\Gamma_2', \Pi_2' \rhd \Delta_2'), \\ u(k) & \text{if } k \in L(G'), \end{cases}$$

and then obtain

$$\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma_1', \Pi_1' \rhd \Delta_1')^r \wedge (G' \mid \Gamma_2', \Pi_2' \rhd \Delta_2')^s,$$

that is

$$\vdash_{\mathcal{S}_{CS}} (G')^r \vee (G')^s \vee ((\Gamma_1', \Pi_1' \rhd \Delta_1')^r \wedge (\Gamma_2', \Pi_2' \rhd \Delta_2')^s).$$

By Lemma 46, we derive

$$\vdash_{\mathcal{S}_{CS}} (G')^r\sigma \vee (G')^s\sigma \vee ((\Gamma_1', \Pi_1' \rhd \Delta_1')^{\sigma\circ r} \wedge (\Gamma_2', \Pi_2' \rhd \Delta_2')^{\sigma\circ s}).$$

Thus, we get (by definition of t) that

$$\vdash_{\mathcal{S}_{CS}} (G')^t \vee ((\Gamma_1', \Pi_1' \rhd \Delta_1')^t \wedge (\Gamma_2', \Pi_2' \rhd \Delta_2')^t),$$

and hence, by propositional reasoning in $\mathcal{G}$, we derive

$$\vdash_{\mathcal{S}_{CS}} (G')^t \vee (\Gamma_1', \Gamma_2' \rhd \Delta_1')^t \vee (\Pi_1', \Pi_2' \rhd \Delta_2')^t$$

as, again by propositional reasoning in $\mathcal{G}$, we have

$$\begin{aligned} \vdash_{\mathcal{S}_{CS}} & ((\phi_1 \wedge \psi_1 \to \chi_1) \wedge (\phi_2 \wedge \psi_2 \to \chi_2)) \\ & \to ((\phi_1 \wedge \phi_2 \to \chi_1) \vee (\psi_1 \wedge \psi_2 \to \chi_2)) \end{aligned}$$

as a theorem (even of $\mathcal{G}$) using the axiom scheme $(\phi \to \psi) \vee (\psi \to \phi)$.

The reasoning for normality of t is similar to the normality proofs occurring in the proof of the Realization Merging Theorem: let $2k \in L(H') = \mathrm{dom}(t)$. Then naturally $t(2k) = u(2k) = x_k$ if $2k \in L(G')$ as u is normal. Suppose $2k \in L(\Gamma_1', \Pi_1' \rhd \Delta_1')$. Then $r(2k) = x_k$ as r is normal. Further, we have $2k \notin L(G')$ as H' is properly annotated and by item (2) of the Realization Merging Theorem for (u, σ) we have $\sigma(x_k) = x_k$. Thus $t(2k) = r(2k)\sigma = x_k$. If we assume $2k \in L(\Gamma_2', \Pi_2' \rhd \Delta_2')$, we get this result using similar reasoning. ☐

9.2. **Logical rules.** The only logical rules which we consider explicitly are the branching rules, i.e. those rules which have multiple assumptions, as the handling of them makes essential use of the merging of realizations (as before with (COM)).

Lemma 51 (($\to \rhd$)-case)**.** *Let $H' := G' \mid \Gamma_1', \Gamma_2', \phi' \to \psi' \rhd \Delta'$ be a p.a. hypersequent and suppose there are normal realizations r and s for $G' \mid \Gamma_1' \rhd \phi'$ and $G' \mid \Gamma_2', \psi' \rhd \Delta'$, respectively, such that*

$$\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma_1' \rhd \phi')^r \text{ and } \vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma_2', \psi' \rhd \Delta')^s.$$

Then, there is a normal realization t for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^t$.

Proof. Suppose that $\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma_1' \rhd \phi')^r$ and $\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma_2', \psi' \rhd \Delta')^s$. By the Realization Merging Theorem, we obtain that there exists a normal realization u for G' and a substitution σ such that u and σ hereditarily merge r and s on G'. In particular, we again have

$$(\dagger) \qquad \vdash_{\mathcal{S}_{CS}} (G')^r\sigma \to (G')^u \text{ and } \vdash_{\mathcal{S}_{CS}} (G')^s\sigma \to (G')^u.$$

By propositional reasoning, we thus derive

$$\vdash_{\mathcal{S}_{CS}} (G')^r \vee (G')^s \vee ((\Gamma_1' \rhd \phi')^r \wedge (\Gamma_2', \psi' \rhd \Delta')^s)$$

and by the Substitution Lemma, we get

$$\vdash_{\mathcal{S}_{CS}} (G')^r\sigma \vee (G')^s\sigma \vee ((\Gamma_1' \rhd \phi')^{\sigma\circ r} \wedge (\Gamma_2', \psi' \rhd \Delta')^{\sigma\circ s}).$$

Propositionally, with (†), this implies

$$\vdash_{\mathcal{S}_{CS}} (G')^u \vee \left(\left(\bigwedge \Gamma_1'\right)^{\sigma\circ r} \wedge \left(\bigwedge \Gamma_2'\right)^{\sigma\circ s} \wedge ((\phi')^{\sigma\circ r} \to (\psi')^{\sigma\circ s}) \to \left(\bigvee \Delta'\right)^{\sigma\circ s}\right).$$

Defining t by

$$t(k) := \begin{cases} r(k)\sigma & \text{if } k \in L(\Gamma_1' \rhd \phi'), \\ s(k)\sigma & \text{if } k \in L(\Gamma_2', \psi' \rhd \Delta'), \\ u(k) & \text{if } k \in L(G'). \end{cases}$$

then results in $\vdash_{\mathcal{S}_{CS}} (H')^t$. Now, t can be shown to be normal by similar reasoning as with the rule (COM). ☐

Lemma 52 (($\rhd\wedge$)-case)**.** *Let $H' := G' \mid \Gamma' \rhd \phi' \wedge \psi'$ be a p.a. hypersequent and suppose there are normal realizations r and s for $G' \mid \Gamma' \rhd \phi'$ and $G' \mid \Gamma' \rhd \psi'$, respectively, such that*

$$\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma' \rhd \phi')^r \text{ and } \vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma' \rhd \psi')^s.$$

Then, there is a normal realization t for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^t$.

Proof. Suppose $\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma' \rhd \phi')^r$ and $\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma' \rhd \psi')^s$. Then, by the Realization Merging Theorem, there exists a normal realization u for $G' \mid \Gamma' \rhd$ and a substitution σ which hereditarily merge r, s on $G' \mid \Gamma' \rhd$. That is, we in particular have:

$$(*) \begin{cases} \vdash_{\mathcal{S}_{CS}} (G')^r\sigma \to (G')^u; \\ \vdash_{\mathcal{S}_{CS}} (G')^s\sigma \to (G')^u; \end{cases}$$

$$(**) \begin{cases} \vdash_{\mathcal{S}_{CS}} (\bigwedge \Gamma')^u \to (\bigwedge \Gamma')^r \sigma; \\ \vdash_{\mathcal{S}_{CS}} (\bigwedge \Gamma')^u \to (\bigwedge \Gamma')^s \sigma. \end{cases}$$

Therefore, by propositional reasoning, we obtain the following implications:

$$\begin{aligned} &\vdash_{\mathcal{S}_{CS}} (G')^r \vee (G')^s \vee ((\Gamma' \rhd \phi')^r \wedge (\Gamma' \rhd \psi')^s) \\ &\quad \text{implies } \vdash_{\mathcal{S}_{CS}} (G')^u \vee \left(\left(\bigwedge \Gamma'\right)^r \sigma \to (\phi')^{\sigma\circ r} \wedge \left(\bigwedge \Gamma'\right)^s \sigma \to (\psi')^{\sigma\circ s}\right) \text{ by } (*) \\ &\quad \text{implies } \vdash_{\mathcal{S}_{CS}} (G')^u \vee \left(\left(\bigwedge \Gamma'\right)^u \to (\phi')^{\sigma\circ r} \wedge \left(\bigwedge \Gamma'\right)^u \to (\psi')^{\sigma\circ s}\right) \text{ by } (**) \\ &\quad \text{implies } \vdash_{\mathcal{S}_{CS}} (G')^u \vee \left(\left(\bigwedge \Gamma'\right)^u \to (\phi')^{\sigma\circ r} \wedge (\psi')^{\sigma\circ s}\right). \end{aligned}$$

Again, we define t by

$$t(k) := \begin{cases} r(k)\sigma & \text{if } k \in L(\phi'), \\ r(k)\sigma & \text{if } k \in L(\psi'), \\ u(k) & \text{if } k \in L(G' \mid \Gamma' \rhd), \end{cases}$$

and then derive $\vdash_{\mathcal{S}_{CS}} (H')^t$. As before, normality of t follows as with the (COM)-rule. ☐

Lemma 53 (($\vee\rhd$)-case)**.** *Let $H' := G' \mid \Gamma', \phi' \vee \psi' \rhd \Delta'$ be a p.a. hypersequent and suppose there are normal realizations r and s for $G' \mid \Gamma', \phi' \rhd \Delta'$ and $G' \mid \Gamma', \psi' \rhd \Delta'$, respectively, such that*

$$\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma', \phi' \rhd \Delta')^r \text{ and } \vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma', \psi' \rhd \Delta')^s.$$

Then, there is a normal realization t for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^t$.

Proof. Suppose $\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma', \phi' \rhd \Delta')^r$ and $\vdash_{\mathcal{S}_{CS}} (G' \mid \Gamma', \psi' \rhd \Delta')^s$. By the Realization Merging Theorem, there is a normal realization u for $G' \mid \Gamma' \rhd \Delta'$ together with a substitution σ which hereditarily merge r, s on $G' \mid \Gamma' \rhd \Delta'$. That is, we again in particular have

$$\begin{aligned} &\vdash_{\mathcal{S}_{CS}} (G')^r\sigma \to (G')^u, && \vdash_{\mathcal{S}_{CS}} (G')^s\sigma \to (G')^u, \\ &\vdash_{\mathcal{S}_{CS}} \left(\bigwedge \Gamma'\right)^u \to \left(\bigwedge \Gamma'\right)^r \sigma, && \vdash_{\mathcal{S}_{CS}} \left(\bigwedge \Gamma'\right)^u \to \left(\bigwedge \Gamma'\right)^s \sigma, \\ &\vdash_{\mathcal{S}_{CS}} \left(\bigvee \Delta'\right)^r \sigma \to \left(\bigvee \Delta'\right)^u, && \vdash_{\mathcal{S}_{CS}} \left(\bigvee \Delta'\right)^s \sigma \to \left(\bigvee \Delta'\right)^u. \end{aligned}$$

As before, using propositional reasoning over $\mathcal{G}$, we get

$$\vdash_{\mathcal{S}_{CS}} (G')^r \vee (G')^s \vee ((\Gamma', \phi' \rhd \Delta')^r \wedge (\Gamma', \psi' \rhd \Delta')^s)$$

by the assumptions and therefore, using the Substitution Lemma, this yields

$$\vdash_{\mathcal{S}_{CS}} (G')^u \vee \left(\left(\bigwedge \Gamma'\right)^u \wedge ((\phi')^{\sigma\circ r} \vee (\psi')^{\sigma\circ s}) \to \left(\bigvee \Delta'\right)^u\right).$$

with additional propositional reasoning and together with the above implications from the Realization Merging Theorem.

We define t by

$$t(k) := \begin{cases} r(k)\sigma & \text{if } k \in L(\phi'), \\ s(k)\sigma & \text{if } k \in L(\psi'), \\ u(k) & \text{if } k \in L(G' \mid \Gamma' \rhd \Delta'), \end{cases}$$

and obtain $\vdash_{\mathcal{S}_{CS}} (H')^t$ from the last line. Normality of t can again be established by similar reasoning as with the rule (COM). ☐

9.3. **Modal rules.** At last, we consider the various modal rules. As they require reasoning using actual principles of justifications in the Gödel justification calculi, we give full proofs in each case.

Lemma 54 (($\Box^-$)-case)**.** *Let* $G' := \gamma'_1, \dots, \gamma'_n \rhd \phi'$, $H' := \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \Box_k \phi'$ *be p.a. and let* r *be a normal realization for* G' *such that* $\vdash_{\mathcal{S}_{CS}} (G')^r$. *Then, there is a normal realization* s *for* H' *such that* $\vdash_{\mathcal{S}_{CS}} (H')^s$.

Proof. Let G' and H' be given as required where $\vdash_{\mathcal{S}_{CS}} (G')^r$ for a normal realization r for G'. We thus have

$$(\dagger) \qquad \vdash_{\mathcal{S}_{CS}} \bigwedge_{j=1}^{n} (\gamma')_j^r \to (\phi')^r.$$

We set $s(m) := r(m)$ for all $m \in L(G')$. As H' is properly annotated, we get $i_j \notin L(G')$ for all $1 \leq j \leq n$. We hence set $s(i_j) := x_{(i_j)/2}$ for all $j \in \{1, \dots, n\}$ which results in a well-defined and normal realization (so far) as H' is properly annotated and (thus) all i_j are even.

Now, by the Lifting Lemma (see also Remark 5), we have by ($\dagger$) that there is a term $t \in Jt$ such that

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{j=1}^{n} s(i_j) : (\gamma'_j)^r \to t : (\phi')^r.$$

Thus, by setting $s(k) := t$, we then immediately obtain that s is a normal realization for H' with $\vdash_{\mathcal{S}_{CS}} (H')^s$ as $s \restriction_{L(G')} = r \restriction_{L(G')}$.[9] $\Box$

Before moving on to the rule ($\Box$), we give the following preliminary lemma.

For this, let S be the class of Gödel-Mkrtychev models with respect to which completeness was proved for $\mathcal{S}_0$.

Lemma 55. *Let* Γ, Δ *be finite sets of justification formulae (in the appropriate language for* $\mathcal{S}_0$*) and* ϕ *be a formula. Let* CS *be* any *constant specification for* $\mathcal{S}_0$. *We have*

$$\vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{\delta \in \Delta} \delta \to \bot \right) \vee \left(\bigwedge_{\gamma \in \Gamma} \gamma \to \phi \right)$$

if, and only if

$$\neg\neg\Delta, \Gamma \vdash_{\mathcal{S}_{CS}} \phi$$

where $\neg\neg\Delta := \{\neg\neg\delta \mid \delta \in \Delta\}$.

Proof. By the Completeness Theorem for S_{CS}-models, Theorem 3, we have

$$\vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{\delta \in \Delta} \delta \to \bot \right) \vee \left(\bigwedge_{\gamma \in \Gamma} \gamma \to \phi \right) \text{ iff } \models_{\mathsf{S}_{\mathsf{CS}}} \left(\bigwedge_{\delta \in \Delta} \delta \to \bot \right) \vee \left(\bigwedge_{\gamma \in \Gamma} \gamma \to \phi \right)$$

and, by additionally using the Deduction Theorem, we obtain

$$\neg\neg\Delta, \Gamma \vdash_{\mathcal{S}_{CS}} \phi \text{ iff } \neg\neg\Delta \vdash_{\mathcal{S}_{CS}} \bigwedge_{\gamma \in \Gamma} \gamma \to \phi \text{ iff } \neg\neg\Delta \models_{\mathsf{S}_{\mathsf{CS}}} \bigwedge_{\gamma \in \Gamma} \gamma \to \phi.$$

[9]We write $f \restriction_Y$ for the restriction of a function $f : X \to Z$ to a subdomain $Y \subseteq X$.

We get the following equivalences:

$$\begin{aligned}
&\models_{\mathsf{S_{CS}}} \left(\bigwedge_{\delta\in\Delta}\delta\to\bot\right)\vee\left(\bigwedge_{\gamma\in\Gamma}\gamma\to\phi\right)\\
&\quad\text{iff } \forall\mathfrak{M}\in\mathsf{S_{CS}} : \max\left\{|\bigwedge_{\delta\in\Delta}\delta\to\bot|_{\mathfrak{M}}, |\bigwedge_{\gamma\in\Gamma}\gamma\to\phi|_{\mathfrak{M}}\right\} = 1\\
&\quad\text{iff } \forall\mathfrak{M}\in\mathsf{S_{CS}} : |\bigwedge_{\delta\in\Delta}\delta|_{\mathfrak{M}} = 0 \text{ or } |\bigwedge_{\gamma\in\Gamma}\gamma|_{\mathfrak{M}} \leq |\phi|_{\mathfrak{M}}\\
&\quad\text{iff } \forall\mathfrak{M}\in\mathsf{S_{CS}} : |\bigwedge_{\delta\in\Delta}\delta|_{\mathfrak{M}} > 0 \text{ implies } |\bigwedge_{\gamma\in\Gamma}\gamma|_{\mathfrak{M}} \leq |\phi|_{\mathfrak{M}}\\
&\quad\text{iff } \forall\mathfrak{M}\in\mathsf{S_{CS}} : |\bigwedge_{\delta\in\Delta}\neg\neg\delta|_{\mathfrak{M}} = 1 \text{ implies } |\bigwedge_{\gamma\in\Gamma}\gamma|_{\mathfrak{M}} \leq |\phi|_{\mathfrak{M}}\\
&\quad\text{iff } \forall\mathfrak{M}\in\mathsf{S_{CS}} : |\neg\neg\Delta|_{\mathfrak{M}} = 1 \text{ implies } |\bigwedge_{\gamma\in\Gamma}\gamma|_{\mathfrak{M}} \leq |\phi|_{\mathfrak{M}}\\
&\quad\text{iff } \neg\neg\Delta \models_{\mathsf{S_{CS}}} \bigwedge_{\gamma\in\Gamma}\gamma\to\phi.
\end{aligned}$$

□

Lemma 56 (($\Box$)-case)**.** *Assume $S_0 \in \{\mathcal{PGJ}_0, \mathcal{PGJT}_0, \mathcal{PGJ}4_0, \mathcal{PGLP}_0\}$ and let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Let*

$$H' := \Box_{j_1}\pi'_1, \dots, \Box_{j_m}\pi'_m \rhd \mid \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \Box_k\phi'$$

be a p.a. hypersequent and let r be a normal realization for

$$G' := \pi'_1, \dots, \pi'_m \rhd \mid \gamma'_1, \dots, \gamma'_n \rhd \phi'$$

such that $\vdash_{\mathcal{S}_{CS}} (G')^r$. Then, there is a normal realization s for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^s$.

Proof. By hypothesis, we have $\vdash_{\mathcal{S}_{CS}} (G')^r$, i.e. spelled out we have

$$\vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{l=1}^{m}(\pi'_l)^r \to \bot\right) \vee \left(\bigwedge_{l=1}^{n}(\gamma'_l)^r \to (\phi')^r\right).$$

By Lemma 55, this implies

$$(\dagger)\qquad \neg\neg(\pi'_1)^r, \dots, \neg\neg(\pi'_m)^r, (\gamma'_1)^r, \dots, (\gamma'_n)^r \vdash_{\mathcal{S}_{CS}} (\phi')^r.$$

We set $s(h) := r(h)$ for $h \in L(G')$ and $s(i_l) := x_{(i_l)/2}$ as well as $s(j_l) := x_{(j_l)/2}$ which is again (so far) well-defined and normal as H' is properly annotated and (therefore) the i_l, j_l are even. Now we have, by the Lifting Lemma (Remark 5) and by (†), that

$$\vartheta s(j_1) : \neg\neg(\pi'_1)^r, \dots, \vartheta s(j_m) : \neg\neg(\pi'_m)^r, s(i_1) : (\gamma'_1)^r, \dots, s(i_n) : (\gamma'_n)^r \vdash_{\mathcal{S}_{CS}} t : (\phi')^r$$

for some $t \in Jt_\vartheta$. We then set $s(k) := t$ and by the axiom scheme (P), we derive

$$\vdash_{\mathcal{S}_{CS}} \neg\neg s(j_l) : (\pi'_l)^r \to \vartheta s(j_l) : \neg\neg(\pi'_l)^r$$

for all $l = 1, \dots, m$ which implies

$$\neg\neg s(j_1) : (\pi'_1)^r, \dots, \neg\neg s(j_m) : (\pi'_m)^r, s(i_1) : (\gamma'_1)^r, \dots, s(i_n) : (\gamma'_n)^r \vdash_{\mathcal{S}_{CS}} t : (\phi')^r$$

using propositional reasoning. Again by Lemma 55, this yields

$$(\ddagger)\qquad \vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{l=1}^{m} s(j_l) : (\pi'_l)^r \to \bot\right) \vee \left(\bigwedge_{l=1}^{n} s(i_l) : (\gamma'_l)^r \to t : (\phi')^r\right)$$

It is straightforward to see that (‡) is equivalent with $\vdash_{\mathcal{S}_{CS}} (H')^s$ and as r is normal, s is normal by construction. □

Lemma 57 (($\Box\rhd$)-case)**.** *Assume $S_0 \in \{\mathcal{GJT}_0, \mathcal{GLP}_0, \mathcal{PGJT}_0, \mathcal{PGLP}_0\}$ and let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Let $G' \mid \Box_k\phi', \Gamma' \rhd \Delta'$ be a p.a. hypersequent and suppose there is a normal realization r for $G' \mid \phi', \Gamma' \rhd \Delta'$ such that*

$$\vdash_{\mathcal{S}_{CS}} (G' \mid \phi', \Gamma' \rhd \Delta')^r.$$

Then, there is a normal realization s for $G' \mid \Box_k\phi', \Gamma' \rhd \Delta'$ such that

$$\vdash_{\mathcal{S}_{CS}} (G' \mid \Box_k\phi', \Gamma' \rhd \Delta')^s.$$

Proof. Set $H' := G' \mid \phi', \Gamma' \rhd \Delta'$ and $I' := G' \mid \Box_k\phi', \Gamma' \rhd \Delta'$. Then, by assumption, we have $\vdash_{\mathcal{S}_{CS}} (H')^r$ for a normal realization r. This translates to

$$\vdash_{\mathcal{S}_{CS}} (G')^r \vee \left(\left((\phi')^r \wedge \left(\bigwedge \Gamma'\right)^r\right) \to \left(\bigvee \Delta'\right)^r\right).$$

We set $s(i) := r(i)$ for $i \in L(H')$ and $s(k) := x_{k/2}$ (note that k is even as it is a negative index in I' and I' is p.a.).

As $\vdash_{\mathcal{S}_{CS}} t : (\phi')^r \to (\phi')^r$, by axiom (F), for all t, we have by propositional reasoning in $\mathcal{S}_{CS}$ that

$$\vdash_{\mathcal{S}_{CS}} (G')^r \vee \left(\left(s(k) : (\phi')^r \wedge \left(\bigwedge \Gamma'\right)^r\right) \to \left(\bigvee \Delta'\right)^r\right).$$

Clearly s is normal by construction as r is normal and also $\vdash_{\mathcal{S}_{CS}} (H')^s$. $\Box$

Lemma 58 (($\rhd\Box^-$)$_1$-case)**.** *Assume $S_0 \in \{\mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{PGJ}4_0, \mathcal{PGLP}_0\}$ and let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Let*

$$H' := \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \Box_k\phi'$$

be a p.a. hypersequent. Let r be a normal realization for

$$G' := \gamma'_1, \dots, \gamma'_n, \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \phi'$$

such that $\vdash_{\mathcal{S}_{CS}} (G')^r$. Then, there is a normal realization s for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^s$.

Proof. $\vdash_{\mathcal{S}_{CS}} (G')^r$ translates to

$$(\dagger) \qquad \vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{n} (\gamma'_l)^r \wedge \bigwedge_{l=1}^{n} x_{j_l} : (\gamma'_l)^r \to (\phi')^r$$

as r is normal and where $r(i_l) = x_{j_l}$. By the Lifting Lemma, there is a term t such that

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{n} x_{j_l} : (\gamma'_l)^r \wedge \bigwedge_{l=1}^{n} !x_{j_l} : x_{j_l} : (\gamma'_l)^r \to t : (\phi')^r$$

and using the axiom scheme (!), we derive

$$\vdash_{\mathcal{S}_{CS}} x_{j_k} : (\gamma'_k)^r \to !x_{j_k} : x_{j_k} : (\gamma'_k)^r.$$

Thus, the above implies

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{n} x_{j_l} : (\gamma'_l)^r \to t : (\phi')^r$$

which gives the claim if we set

$$s(m) := \begin{cases} r(m) & \text{if } m \in L(G'), \\ t & \text{if } m = k, \end{cases}$$

for $m \in L(H')$ using the respective t, as then $\vdash_{\mathcal{S}_{CS}} (H')^s$ and s is clearly normal. $\Box$

Lemma 59 (($\rhd\Box^-$)$_2$-case)**.** *Assume $S_0 \in \{\mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{PGJ}4_0, \mathcal{PGLP}_0\}$ and let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Let*

$$H' := \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \Box_k\phi'$$

be a p.a. hypersequent. Let r be a normal realization for

$$G' := \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \phi'$$

such that $\vdash_{\mathcal{S}_{CS}} (G')^r$. Then, there is a normal realization s for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^s$.

Proof. Spelled out, $\vdash_{\mathcal{S}_{CS}} (G')^r$ is exactly

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{n} x_{j_l} : (\gamma_l')^r \to (\phi')^r$$

with $r(i_l) = x_{j_l}$. By propositional reasoning, we obtain

$$\vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{l=1}^{n} (\gamma_l')^r \wedge \bigwedge_{l=1}^{n} x_{j_l} : (\gamma_l')^r \right) \to (\phi')^r.$$

For $I' := \gamma_1', \ldots, \gamma_n', \Box_{i_1}\gamma_1', \ldots, \Box_{i_n}\gamma_n' \rhd \phi'$, that is exactly $\vdash_{\mathcal{S}_{CS}} (I')^r$ and Lemma 58 gives the result. □

Lemma 60 (($\rhd\Box)_1$-case)**.** *Assume $S_0 \in \{\mathcal{PGJ}4_0, \mathcal{PGLP}_0\}$ and let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Let*

$$H' := \Box_{j_1}\pi_1', \ldots, \Box_{j_m}\pi_m' \rhd \mid \Box_{i_1}\gamma_1', \ldots, \Box_{i_n}\gamma_n' \rhd \Box_k\phi'$$

be a p.a. hypersequent. Let r be a normal realization for

$$G' := \pi_1', \ldots, \pi_m', \Box_{j_1}\pi_1', \ldots, \Box_{j_m}\pi_m' \rhd \mid \gamma_1', \ldots, \gamma_n', \Box_{i_1}\gamma_1', \ldots, \Box_{i_n}\gamma_n' \rhd \phi'$$

such that $\vdash_{\mathcal{S}_{CS}} (G')^r$. Then, there is a normal realization s for H' such that $\vdash_{\mathcal{S}_{CS}} (H')^s$.

Proof. Suppose that there is an r such that $\vdash_{\mathcal{S}_{CS}} (G')^r$. Then, by unwinding the realization and applying Lemma 55, we obtain

$$\bigwedge_{l=1}^{m} \neg\neg(\pi_l')^r, \bigwedge_{l=1}^{m} \neg\neg r(j_l) : (\pi_l')^r, \bigwedge_{l=1}^{n} (\gamma_l')^r, \bigwedge_{l=1}^{n} r(i_l) : (\gamma_l')^r \vdash_{\mathcal{S}_{CS}} (\phi')^r.$$

By the Lifting Lemma, there is a term $t \in Jt$ such that

$$\bigwedge_{l=1}^{m} \vartheta r(j_l) : \neg\neg(\pi_l')^r, \bigwedge_{l=1}^{m} \vartheta ! r(j_l) : \neg\neg r(j_l) : (\pi_l')^r,$$

$$(\dagger) \qquad \bigwedge_{l=1}^{n} r(i_l) : (\gamma_l')^r, \bigwedge_{l=1}^{n} ! r(i_l) : r(i_l) : (\gamma_l')^r \vdash_{\mathcal{S}_{CS}} t : (\phi')^r.$$

By the axiom scheme (!), we obtain

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{n} r(i_l) : (\gamma_l')^r \to \bigwedge_{l=1}^{n} ! r(i_l) : r(i_l) : (\gamma_l')^r.$$

Further, by axiom scheme (P), we have

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{m} \neg\neg r(j_l) : (\pi_l')^r \to \bigwedge_{l=1}^{m} \vartheta r(j_l) : \neg\neg(\pi_l')^r$$

as well as

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{m} \neg\neg ! r(j_l) : r(j_l) : (\pi_l')^r \to \bigwedge_{l=1}^{m} \vartheta ! r(j_l) : \neg\neg r(j_l) : (\pi_l')^r.$$

Using the axiom scheme (!) as well as the propositional theorem

$$\vdash_{\mathcal{S}_{CS}} (\phi \to \psi) \to (\neg\neg\phi \to \neg\neg\psi),$$

we get

$$\vdash_{\mathcal{S}_{CS}} \bigwedge_{l=1}^{m} \neg\neg r(j_l) : (\pi_l')^r \to \bigwedge_{l=1}^{m} \neg\neg ! r(j_l) : r(j_l) : (\pi_l')^r.$$

Hence, (†) reduces to

$$(\ddagger) \qquad \bigwedge_{l=1}^{m} \neg\neg r(j_l) : (\pi_l')^r, \bigwedge_{l=1}^{n} r(i_l) : (\gamma_l')^r \vdash_{\mathcal{S}_{CS}} t : (\phi')^r.$$

The desired realization s can be given through

$$s(h) := \begin{cases} r(h) & \text{if } h \in L(G'), \\ t & \text{if } h = k, \end{cases}$$

where $h \in L(H')$. Then, (‡) is exactly $\vdash_{\mathcal{S}_{CS}} (H')^s$. This s can also easily be seen to be normal. □

Lemma 61 (($(\rhd\Box)_2$-case)**.** *Assume* $S_0 \in \{\mathcal{PGJ}4_0, \mathcal{PGLP}_0\}$ *and let* CS *be a schematic and axiomatically appropriate constant specification for* $\mathcal{S}_0$*. Let*

$$H' := \Box_{j_1}\pi'_1, \dots, \Box_{j_m}\pi'_m \rhd \mid \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \Box_k\phi'$$

be a p.a. hypersequent. Let r *be a normal realization for*

$$G' := \Box_{j_1}\pi'_1, \dots, \Box_{j_m}\pi'_m \rhd \mid \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \phi'$$

such that $\vdash_{\mathcal{S}_{CS}} (G')^r$*. Then, there is a normal realization* s *for* H' *such that* $\vdash_{\mathcal{S}_{CS}} (H')^s$*.*

Proof. As in the proof of Lemma 59, we have

$$\vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{l=1}^{m} r(j_l) : (\pi'_l)^r \to \bot\right) \vee \left(\bigwedge_{l=1}^{n} r(i_l) : (\gamma'_l)^r \to (\phi')^r\right)$$

from $\vdash_{\mathcal{S}_{CS}} (G')^r$ and, by propositional reasoning, we derive

$$\vdash_{\mathcal{S}_{CS}} \left(\bigwedge_{l=1}^{m}(\pi'_l)^r \wedge \bigwedge_{l=1}^{m} r(j_l) : (\pi'_l)^r \to \bot\right) \vee \left(\bigwedge_{l=1}^{m}(\gamma'_l)^r \wedge \bigwedge_{l=1}^{n} r(i_l) : (\gamma'_l)^r \to (\phi')^r\right)$$

After setting $I' := \pi'_1, \dots, \pi'_m, \Box_{j_1}\pi'_1, \dots, \Box_{j_m}\pi'_m \rhd \mid \gamma'_1, \dots, \gamma'_n, \Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n \rhd \phi'$, that amounts to $\vdash_{\mathcal{S}_{CS}}$ $(I')^r$ and thus the result follows by Lemma 60. □

We now turn to the main result.

Theorem 62 (Realization Theorem)**.** *Let*

$$\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0\}$$

and

$$\mathcal{GML}_\Box \in \{\mathcal{GK}_\Box, \mathcal{GT}_\Box, \mathcal{GK}4_\Box, \mathcal{GS}4_\Box\}.$$

be the respective corresponding modal logic. Let CS, CS' *be schematic and axiomatically appropriate constant specifications for* $\mathcal{GJL}_0, \mathcal{PGJL}_0$*, respectively. Then*

$$Th_{\mathcal{GML}_\Box^-} = (Th_{\mathcal{GJL}_{CS}})^\nu \text{ and } Th_{\mathcal{GML}_\Box} = (Th_{\mathcal{PGJL}_{CS'}})^\nu.$$

Proof. The inclusions from right to left come from Lemma 21 and Lemma 33. Let $\mathcal{GML}_\Box^\pm$ be the modal calculus and $\mathcal{S}_0$ be the (maybe positive) justification calculus. Further, let CS be a schematic and axiomatically appropriate constant specification for $\mathcal{S}_0$. Corresponding to $\mathcal{GML}_\Box^\pm$, let $\mathcal{HGML}_\Box^\pm$ be the corresponding hypersequent calculus and $\mathcal{HGML}_\Box^{\pm\prime}$ be its annotated version as before.

To establish the theorem, we first show the following claim:

Claim: If $\vdash_{\mathcal{HGML}_\Box^{\pm\prime}} G'$ where G' is properly annotated, then there is a normal realization r for G' such that $\vdash_{\mathcal{S}_{CS}} (G')^r$.

Proof: By induction on the length of the proof in $\mathcal{HGML}_\Box^{\pm\prime}$. For the initial hypersequents (ID^w), ($\rhd\top^\mathsf{w}$), ($\bot\rhd^\mathsf{w}$), the empty realization suffices which is also, naturally, normal.

For the induction step, suppose the claim holds for all properly annotated hypersequents H' with a proof of length $\leq k$ and let G' have a proof of length $k+1$.

Now, if G' is an initial hypersequent, then the empty realization suffices again. If G' is obtained by any of the rules *but* (EC), (CL), $(\rhd\Box^-)'_1$ and $(\rhd\Box)'_1$, the premise(s) are properly annotated. By the induction hypothesis, as the proof(s) are shorter, there are normal realizations for them. For most of the rules, it simply suffices to carry over these previous realizations to the conclusion G'. We thus only mention the following interesting cases (where more is needed).

For the branching rules (COM), $(\to\rhd)$, $(\rhd\wedge)$ and $(\vee\rhd)$, the required realization for G' can be obtained through the Lemmas 50, 51, 52 and 53, respectively.

For the modal rules $(\Box^-)'$, $(\Box)'$, $(\Box\rhd)'$, $(\rhd\Box^-)'_2$ and $(\rhd\Box)'_2$, the required realization for G' can be obtained through the Lemmas 54, 56, 57, 59 and 61, respectively.

Now, if G' is obtained by either (1) (EC), (2) (CL), (3) $(\rhd\Box)'_1$ or (4) $(\rhd\Box^-)'_1$, then the premise is not properly annotated so we handle these cases explicitly.

(1) We have $G' = H'_1 \mid H'_2$ with properly annotated hypersequents H'_1, H'_2. We write $H_j = (H'_j)^\bullet$ for $j = 1, 2$. As G' was obtained by (EC), the annotated hypersequent $H'_1 \mid H'_2 \mid H'_2$, although not properly annotated anymore, is provable with a shorter proof. However, we may consider a different properly annotated hypersequent H''_2 with $(H''_2)^\bullet = H_2$ such that, additionally, $H'_1 \mid H'_2 \mid H''_2$ is properly annotated. By

Lemma 49, this *reannotation* of $H_1' \mid H_2' \mid H_2'$ has a proof of the same length. By the induction hypothesis, there is a normal realization function s for $H_1' \mid H_2' \mid H_2''$ such that $\vdash_{\mathcal{S}_{CS}} (H_1' \mid H_2' \mid H_2'')^s$.

We set $s_1 := s \restriction_{L(H_2')}$ and $s_2 := \left(s \restriction_{L(H_2'')}\right) \circ L_{H_2',H_2''}$. Note, that also s_2 is a realization with domain $L(H_2')$ (see Definition 22). However, due to the change of labels in s_2, it may not be normal anymore. To reobtain normality, we introduce the following substitution σ':

$$\sigma'(x_m) = \begin{cases} x_n & \text{if } L_{H_2',H_2''}(2n) = 2m \text{ where } 2m \in L(H_2''), \\ x_m & \text{otherwise.} \end{cases}$$

Note that σ' is well-defined as $L_{H_2',H_2''}$ is a bijection and that $\mathrm{dom}(\sigma') = \{x_m \mid 2m \in L(H_2'')\}$. Now, $\sigma' \circ s_1$ and $\sigma' \circ s_2$ are both normal. For the former, as $\mathrm{dom}(\sigma') = \{x_m \mid 2m \in L(H_2'')\}$, we have that $x_m \in \mathrm{dom}(\sigma)'$ implies $2m \notin L(H_2') = \mathrm{dom}(s_1)$ as $H_1' \mid H_2' \mid H_2''$ is properly annotated. Thus $\sigma' \circ s_1$ is normal. For the latter, let $2n \in L(H_2')$ and let m be such that $L_{H_2',H_2''}(2n) = 2m$. Then we have $s_2(2n) = s(2m) = x_m$ by normality of s and thus $(\sigma' \circ s_2)(2n) = x_m\sigma' = x_n$. Therefore $\sigma' \circ s_2$ is normal.

By the Realization Merging Theorem, we obtain that there is a normal realization q for H_2' together with a substitution σ which hereditarily merge $\sigma' \circ s_1$ and $\sigma' \circ s_2$ on H_2'. In particular, this implies

$$\vdash_{\mathcal{S}_{CS}} (H_2')^{\sigma' \circ s_i}\sigma \to (H_2')^q$$

for $i = 1, 2$. We therefore have

$$\begin{aligned} \vdash_{\mathcal{S}_{CS}} (H_1')^s \vee (H_2')^s \vee (H_2'')^s \text{ implies } &\vdash_{\mathcal{S}_{CS}} (H_1')^s \vee (H_2')^{s_1} \vee (H_2')^{s_2} \\ \text{implies } &\vdash_{\mathcal{S}_{CS}} ((H_1')^s \vee (H_2')^{s_1} \vee (H_2')^{s_2})\sigma'\sigma \\ \text{implies } &\vdash_{\mathcal{S}_{CS}} (H_1')^s\sigma'\sigma \vee (H_2')^{\sigma' \circ s_1}\sigma \vee (H_2')^{\sigma' \circ s_2}\sigma \\ \text{implies } &\vdash_{\mathcal{S}_{CS}} (H_1')^{\sigma'\sigma \circ s} \vee (H_2')^q \end{aligned}$$

We take the desired realization to be

$$r(n) := \begin{cases} (\sigma'\sigma \circ s)(n) & \text{if } n \in L(H_1'), \\ q(n) & \text{if } n \in L(H_2'). \end{cases}$$

This r is normal: if $2n \in \mathrm{dom}(r)$, then either $2n \in L(H_2')$ and thus $r(2n) = q(2n) = x_n$ as q is normal or we have $2n \in L(H_1')$. Then, as s is normal we have $s(2n) = x_n$. As $H_1' \mid H_2' \mid H_2''$ is properly annotated, we have $2n \notin L(H_2'')$ and thus $\sigma'(x_n) = x_n$. Further, we also have $2n \notin L(H_2')$. Now, by property (2) of the Realization Merging Theorem and as σ and q hereditarily merge $\sigma' \circ s_1$ and $\sigma' \circ s_2$ on H_2', we have $x_n \notin \mathrm{dom}(\sigma)$ and thus $r(2n) = s(2n)\sigma'\sigma = x_n$.

(2) If G' was obtained by (CL), then $G' = H' \mid \Gamma', \phi' \rhd \Delta'$. As before, we properly annotate $H \mid \Gamma, \phi, \phi \rhd \Delta$ by $H' \mid \Gamma', \phi', \phi'' \rhd \Delta'$ (a reannotation of $H' \mid \Gamma', \phi', \phi' \rhd \Delta'$ in the sense of before) and by the induction hypothesis there is now a normal realization s for it such that

$$\vdash_{\mathcal{S}_{CS}} (H' \mid \Gamma', \phi', \phi'' \rhd \Delta')^s.$$

Now, we define $s_1 := s \restriction_{L(\phi'\rhd)}$ and $s_2 = \left(s \restriction_{L(\phi''\rhd)}\right) \circ L_{(\phi'\rhd),(\phi''\rhd)}$. Again, s_2 fails to be normal due to the change of labels. Thus, we again introduce a substitution σ' defined by

$$\sigma'(x_m) := \begin{cases} x_n & \text{if } L_{(\phi'\rhd),(\phi''\rhd)}(2n) = 2m \text{ where } 2m \in L(\phi''\rhd), \\ x_m & \text{otherwise.} \end{cases}$$

Again, we have that σ' is well-defined as $L_{(\phi'\rhd),(\phi''\rhd)}$ is a bijection and we obtain that $\mathrm{dom}(\sigma') = \{x_m \mid 2m \in L(\phi''\rhd)\}$. Similarly as with the rule (EC), one can show that both $\sigma' \circ s_1$ and $\sigma' \circ s_2$ are normal.

Using the Realization Merging Theorem again, there is a realization q for "$\phi'\rhd$" together with a substitution σ which hereditarily merge $\sigma' \circ s_1$ and $\sigma' \circ s_2$ on "$\phi'\rhd$". Thus, as the $(\phi')^{\sigma' \circ s_i}$ are negative subformulae of $(\phi'\rhd)^{\sigma' \circ s_i}$ for $i = 1, 2$, respectively, this yields

$$\vdash_{\mathcal{S}_{CS}} (\phi')^q \to (\phi')^{\sigma' \circ s_i}\sigma$$

for $i = 1, 2$. Hence, we have

$$\vdash_{\mathcal{S}_{CS}} (H' \mid \Gamma', \phi', \phi'' \rhd \Delta')^s$$
$$\text{implies } \vdash_{\mathcal{S}_{CS}} \left((H')^s \vee \left(\left(\bigwedge \Gamma'\right)^s \wedge (\phi')^{s_1} \wedge (\phi')^{s_2} \to \left(\bigvee \Delta'\right)^s\right)\right)\sigma'\sigma$$
$$\text{implies } \vdash_{\mathcal{S}_{CS}} (H')^s\sigma'\sigma \vee \left(\left(\bigwedge \Gamma'\right)^s \sigma'\sigma \wedge (\phi')^{\sigma' \circ s_1}\sigma \wedge (\phi')^{\sigma' \circ s_2}\sigma \to \left(\bigvee \Delta'\right)^s \sigma'\sigma\right)$$
$$\text{implies } \vdash_{\mathcal{S}_{CS}} (H')^{\sigma'\sigma \circ s} \vee \left(\left(\bigwedge \Gamma'\right)^{\sigma'\sigma\circ s} \wedge (\phi')^q \to \left(\bigvee \Delta'\right)^{\sigma'\sigma\circ s}\right)$$

and get the desired realization from the last line by

$$r(n) := \begin{cases} q(n) & \text{if } n \in L(\phi'), \\ (\sigma'\sigma \circ s)(n) & \text{if } n \in L(G') \setminus L(\phi'). \end{cases}$$

By similar reasoning as with the rule (EC), this r can be shown to be normal.

(3) If G' was obtained by $(\rhd \Box)'_1$, then

$$G' = \Box_{j_1}\pi'_1, \ldots, \Box_{j_m}\pi'_m \rhd \mid \Box_{i_1}\gamma'_1, \ldots, \Box_{i_n}\gamma'_n \rhd \Box_k\phi'$$

and H' defined by

$$H' := \pi'_1, \ldots, \pi'_m, \Box_{j_1}\pi'_1, \ldots, \Box_{j_m}\pi'_m \rhd \mid \gamma'_1, \ldots, \gamma'_n, \Box_{i_1}\gamma'_1, \ldots, \Box_{i_n}\gamma'_n \rhd \phi'$$

is provable with a shorter proof. We properly reannotate H' as H'' with

$$H'' = \pi''_1, \ldots, \pi''_m, \Box_{j_1}\pi'_1, \ldots, \Box_{j_m}\pi'_m \rhd \mid \gamma''_1, \ldots, \gamma''_n, \Box_{i_1}\gamma'_1, \ldots, \Box_{i_n}\gamma'_n \rhd \phi'$$

and by Lemma 49, H'' is provable with a shorter proof than G'. Therefore, by the induction hypothesis, there is a normal realization s for H'' such that

$$(\dagger) \qquad \vdash_{\mathcal{S}_{CS}} (H'')^s.$$

Define

$$H'_1 := \Box_{j_1}\pi'_1, \ldots, \Box_{j_m}\pi'_m \rhd \text{ and } H''_1 := \Box_{j_1}\pi''_1, \ldots, \Box_{j_m}\pi''_m \rhd .$$

and construct s_1, s_2 by $s_1 := s \restriction_{L(H'_1)}$ and $s_2 = \left(s \restriction_{L(H''_1)}\right) \circ L_{H'_1, H''_1}$. Similarly as with the two previous rules, we introduce a substitution σ'_1 given via

$$\sigma'_1(x_m) = \begin{cases} x_n & \text{if } L_{H'_1,H''_1}(2n) = 2m \text{ where } 2m \in L(H''_1), \\ x_m & \text{otherwise,} \end{cases}$$

to make s_2 normal again after the change of labels. It can again be easily seen that σ'_1 is well-defined and that $\mathrm{dom}(\sigma'_1) = \{x_m \mid 2m \in L(H''_1)\}$. From this, normality of $\sigma'_1 \circ s_1$ and $\sigma'_1 \circ s_2$ follows as before.

From the Realization Merging Theorem, we obtain that there is a normal realization q_1 for H'_1 and a substitution σ_1 such that they hereditarily merge $\sigma'_1 \circ s_1$ and $\sigma'_1 \circ s_2$ on H'_1. This in particular (by considering the definitions of s_1, s_2) implies

$$\vdash_{\mathcal{S}_{CS}} (\pi'_l)^{q_1} \to (\pi''_l)^{\sigma'_1 \circ s}\sigma_1,$$
$$\vdash_{\mathcal{S}_{CS}} (\Box_{j_l}\pi'_l)^{q_1} \to (\Box_{j_l}\pi'_l)^{\sigma'_1 \circ s}\sigma_1,$$

as $(\pi'_l)^{s_2} = (\pi''_l)^s$. Similarly, we may define

$$H'_2 := \Box_{i_1}\gamma'_1, \ldots, \Box_{i_n}\gamma'_n \rhd \text{ and } H''_2 := \Box_{i_1}\gamma''_1, \ldots, \Box_{i_n}\gamma''_n \rhd$$

as well as $t_1 := s \restriction_{L(H'_2)}$ and $t_2 := \left(s \restriction_{L(H''_2)}\right) \circ L_{H'_2, H''_2}$. For t_2, we introduce a substitution σ'_2 similar to σ'_1 via

$$\sigma'_2(x_m) = \begin{cases} x_n & \text{if } L_{H'_2,H''_2}(2n) = 2m \text{ where } 2m \in L(H''_2), \\ x_m & \text{otherwise.} \end{cases}$$

One can again show that σ'_2 is well-defined and that $\mathrm{dom}(\sigma'_2) = \{x_m \mid 2m \in L(H''_2)\}$. Normality of $\sigma'_2 \circ t_1$ and $\sigma'_2 \circ t_2$ follows as before. So, again using the Realization Merging Theorem, one obtains a normal realization q_2 for H'_2 and a substitution σ_2 such that they hereditarily merge $\sigma'_2 \circ t_1$ and $\sigma'_2 \circ t_2$ on H'_2. In particular, considering the definition of t_1, t_2, we have

$$\vdash_{\mathcal{S}_{CS}} (\gamma'_l)^{q_2} \to (\gamma''_l)^{\sigma'_2 \circ s}\sigma_2,$$
$$\vdash_{\mathcal{S}_{CS}} (\Box_{i_l}\gamma'_l)^{q_2} \to (\Box_{i_l}\gamma'_l)^{\sigma'_2 \circ s}\sigma_2.$$

Note that $\sigma'_1\sigma_1$ and $\sigma'_2\sigma_2$ commute in the sense of Remark 8 by properties (1) and (2) of the Realization Merging Theorem. Using $\vdash_{\mathcal{S}_{CS}} (H'')^s$, we obtain $\vdash_{\mathcal{S}_{CS}} (H'')^s\sigma'_1\sigma_1\sigma'_2\sigma_2$ by the Substitution Lemma and therefore

$$\vdash_{\mathcal{S}_{CS}} \left(\left(\bigwedge_{l=1}^{m}(\pi''_l)^{\sigma'_1\sigma_1\circ s}\sigma'_2\sigma_2 \wedge \bigwedge_{l=1}^{m}(\Box_{j_l}\pi'_l)^{\sigma'_1\sigma_1\circ s}\sigma'_2\sigma_2\right) \to \bot\right)$$
$$\vee \left(\left(\bigwedge_{l=1}^{n}(\gamma''_l)^{\sigma'_2\sigma_2\circ s}\sigma'_1\sigma_1 \wedge \bigwedge_{l=1}^{n}(\Box_{i_l}\gamma'_l)^{\sigma'_2\sigma_2\circ s}\sigma'_1\sigma_1\right) \to (\phi')^{\sigma'_1\sigma_1\sigma'_2\sigma_2\circ s}\right)$$

Therefore, by the properties of q_1 and q_2 from above, we obtain

$$\vdash_{\mathcal{S}_{CS}} \left(\left(\bigwedge_{l=1}^{m}(\pi'_l)^{\sigma'_2\sigma_2\circ q_1} \wedge \bigwedge_{l=1}^{m}(\Box_{j_l}\pi'_l)^{\sigma'_2\sigma_2\circ q_1}\right) \to \bot\right)$$
$$\vee \left(\left(\bigwedge_{l=1}^{n}(\gamma'_l)^{\sigma'_1\sigma_1\circ q_2} \wedge \bigwedge_{l=1}^{n}(\Box_{i_l}\gamma'_l)^{\sigma'_1\sigma_1\circ q_2}\right) \to (\phi')^{\sigma'_1\sigma_1\sigma'_2\sigma_2\circ s}\right)$$

If we define

$$r'(n) := \begin{cases} (\sigma'_2\sigma_2 \circ q_1)(n) & \text{if } n \in L(\Box_{j_1}\pi'_1, \dots, \Box_{j_m}\pi'_m \rhd), \\ (\sigma'_1\sigma_1 \circ q_2)(n) & \text{if } n \in L(\Box_{i_1}\gamma'_1, \dots, \Box_{i_n}\gamma'_n), \\ (\sigma'_1\sigma_1\sigma'_2\sigma_2 \circ s)(n) & \text{if } n \in L(\phi'). \end{cases}$$

then we obtain that $\vdash_{\mathcal{S}_{CS}} (H')^{r'}$. Again, by similar reasoning as before, this r' is normal. The desired realization r for G' is then given by Lemma 60.

(4) If G' was obtained by $(\rhd\Box^-)'_1$, then this case can be handled in a similar way as the $(\rhd\Box)'_1$-case, using Lemma 58.

■

We then obtain the Realization Theorem as follows: if $\vdash_{\mathcal{GML}^{\pm}_{\Box}} \phi$, then by Theorem 44 and Theorem 45 we have $\vdash_{\mathcal{HGML}^{\pm}_{\Box}} \rhd\phi$. Now, for some proper annotation ϕ' of ϕ, we have that $\vdash_{\mathcal{HGML}^{\pm\prime}_{\Box}} \rhd\phi'$ with an annotated proof by Lemma 49. By the above claim, there is a realization r for ϕ' such that $\vdash_{\mathcal{S}_{CS}} (\phi')^r$. □

10. Discussion

We have shown that the four Gödel justification logics $\mathcal{GJ}_{CS}$, $\mathcal{GJT}_{CS}$, $\mathcal{GJ}4_{CS}$, $\mathcal{GLP}_{CS}$ from [29] do not realize the standard Gödel modal logics $\mathcal{GK}_{\Box}$, $\mathcal{GT}_{\Box}$, $\mathcal{GK}4_{\Box}$ and $\mathcal{GS}4_{\Box}$ from [9] and by this answered one of the open problems from [29] (and, implicitly, from [15]) negatively. In fact, all of them as well as $\mathcal{GJ}45_{CS}$ do not even realize $\mathcal{GK}_{\Box}$ since the problem lies with the axiom scheme (Z).

We didn't consider the Gödel justification logic $\mathcal{GJT}45_{CS}$ (for some constant specification CS) as the methods which were employed to show non-realization for the other logics do not extend to this case. In particular, the model $\mathfrak{M}_x$ is not a GMT45-model, as the factivity condition

$$\mathcal{E}(t,\phi) \leq |\phi|_{\mathfrak{M}}$$

is not satisfied in $\mathfrak{M}_x$ (which actually prompted the alternative semantics from Subsection 4.1). However, the alternative consequence relation for Gödel-Mkrtychev models is sound but not complete with respect to the proof calculus $\mathcal{GJT}45_{CS}$, a phenomenon already occurring in the classical case (see [32]). One might wonder if a suitable Gödel-Fitting model can be found to witness non-realization. However, the idea behind the model $\mathfrak{M}_x$ also does not straightforwardly translate to Gödel-Fitting models, as the GJT45-models have to validate the condition

$$\mathcal{E}(w,t,\phi) \leq |t:\phi|^w_{\mathfrak{M}}$$

in similarity to the above factivity condition which is not straightforward to satisfy in combination with the conditions for the operators ? and !. It shall be interesting to consider $\mathcal{GJT}45_{CS}$ in future work and we conjecture that these non-realizability phenomenons also occur there.

Realization is the core connection between justification logics and modal logics. It is thus feasible to ask, as the Gödel justification logics do not realize the standard Gödel modal logics, as of how the standard Gödel justification logics and not their positive variants are of primary interest. There are, however, reasons for interest in the non-positive versions: the standard Gödel justifications logics can be seen as *natural* generalizations of the classical justification logics in many ways. Firstly, they arise from natural many-valued generalizations of the classical Fitting or Mkrtychev models over the same language. Secondly, they arise by replacing the boolean base of the usual Hilbert-style calculi for the classical justification logics by a calculus for propositional Gödel logic. Further, as we have shown in this paper, they do realize certain Gödel modal logics, the weak Gödel modal logics introduced in this paper. These weak Gödel modal logics, as will be argued in the following subsection, also have a certain natural appeal which makes them an interesting fragment of the standard Gödel modal logics from [9] in their own right. Lastly, the positivity operator introduced into the language of justification terms, to realize the standard Gödel modal logics, does not have such a natural appeal as it is not needed in the classical case.

These considerations are arguments for the point that they are not "the wrong" Gödel justification logics but that there is an effective gap between the standard Gödel justification logics, as natural generalizations of the classical case, and the standard Gödel modal logics, as natural *semantical* generalizations of the classical case. And further, this gap seems to be inherent to the many-valuedness of the base logic, as the semantical approaches to non-realizability in the first part and the origins of the problematic (Z) axiom scheme show. The latter will be discussed in more detail, in combination with the above emphasis on the word *semantical*, later on. Before moving on, we want to note that it will be interesting to see if generalizations of classical justification logics using other t-norm based many-valued logics as base logics share the same results with their corresponding many-valued modal counterparts.

10.1. **Weak Gödel modal logics.** Regarding the weak Gödel modal logics, we first want to acknowledge the similarity of the given semantics over Quasi-Gödel-Kripke models and the definition of Gödel-Fitting models for the Gödel justification logics. Indeed, the controller may be seen as a many-valued evidence function $\mathcal{E}$ restricted to *one* "justification term" which is represented by $\Box$.

Further, we want to mention the merit of the weak Gödel modal logics from a proof-theoretic perspective. The weak Gödel modal logics arise in their Hilbert-style formulation by taking classical Hilbert-style calculi of modal logics and replacing their boolean base by a calculus for Gödel logic. So, in some way, they are faithful *proof-theoretical* generalizations of the classical cases and not *semantical* ones like the standard Gödel modal logics which arise by axiomatizing the theory of the Gödel-Kripke models which are $[0, 1]$-valued generalizations of the classical Kripke structures. As these weak Gödel modal logics diverge from the standard Gödel modal logics from [9], this is also a prime example of how proof-theoretical and semantical generalizations of classical systems may diverge in the context of t-norm based many-valued logics (or intermediate logics) with modal operators.

So, in a way, the (Z) axiom scheme is a product of the chosen approach to many-valued modal logics using $[0, 1]$-valued Kripke models and is debatable from other semantical and proof-theoretical perspectives on modal logics and their generalizations.

10.2. **Positive Gödel justification logics.** The approach to handle the axiom scheme (Z) by the specifically designated new operator ϑ on terms and a corresponding axiom scheme is debatable. There may be other ways of giving an explicit account of (Z), but the motivation for the ϑ-operator comes from the following considerations: the scheme $\phi \to \neg\neg\phi$ is a tautology in the systems based on Gödel logic and thus, for any term t, there is a term s such that $t : \phi \to s : \neg\neg\phi$ is a tautology (utilizing a strong enough constant specification). Now, the version $\neg\neg t : \phi \to s : \neg\neg\phi$ with a doubly negated premise is valid in the context of the law of excluded middle but in general fails to be a tautology as the term s is, by the validity of $t : \phi \to s : \neg\neg\phi$, only guaranteed to be a justification of $\neg\neg\phi$ *to the degree of* t *being a justification for* ϕ, not more. But if $t : \phi$ is evaluated to be positive, then $\neg\neg t : \phi$ is evaluated to be 1.

So the intuition for ϑt is to represent such a *full* justification s for $\neg\neg\phi$ if t is, at any positive degree, a justification for ϕ. The semantics of ϑt reflects this intuition and it shall be interesting to see as to how one can give a different meaning to the positivity operator.

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