ON DECIDABILITY AND COMPLEXITY OF GÖDEL JUSTIFICATION LOGIC

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ABSTRACT. We extend the study of decidability and complexity in justification logics to the non-classical variants with Gödel justification logics. These replace the boolean base of classical justification logics by $[0,1]$-valued propositional Gödel logic, one of the primary examples of both propositional intermediate logics and t-norm based fuzzy logics. Due to the many-valuedness of the underlying propositional part, the considerations are divided between 1-valued and positively-valued satisfiability and validity. We approach these problems model-theoretically, employing a method derived from Hájek's initial investigation into the complexity of t-norm based propositional fuzzy logics. Utilizing this, we show various equivalences between classical and many-valued satisfiability and validity problems for these justification logics.

1. INTRODUCTION

The origin of classical justification logics lies with the so called *logic of proofs*, a propositional modal logic with a family of necessity-style modal operators $t : \phi$, where the index parameter t encodes a *proof* for ϕ in the syntax. This logic was devised by Artemov in [1, 2] to provide the intuitionistic propositional calculus with a classical provability semantics based on formal arithmetic in the sense of the Brouwer-Heyting-Kolmogorov interpretation, following and rediscovering previous ideas and attempts of Gödel in [10, 11]. The interpretation of the terms in the modal operators was later broadened from representing proofs to general justifications and the whole family of these justification logics was lifted into the realm of epistemic logics, coining a subfield of *explicit* epistemic logic. Decidability and complexity considerations for the logic of proofs and its relatives are already present in the earliest papers and even the driving force behind the first non-provability semantics via so called Mkrtychev-models [23]. A first interesting result in these classical cases is that, while the standard classical modal logics based on the usual singular $\Box$-modality are PSPACE-complete (at least as much as the whole polynomial hierarchy), the justification analogues (if decidable) mostly reside in lower ranks of the polynomial hierarchy, that is in particular Π_2^p.

In recent times, there has been a fair interest in various non-classical justification logics and their applications (e.g. in modeling (explicit) uncertain reasoning). Examples for this are in particular many-valued justifications [7, 8, 24] and graded justification logics [22], as well as probabilistic [15, 13, 14], possibilistic justification logics [6] and intuitionistic justification logics [18, 19, 20]. In this paper, we study decidability and complexity issues in a particular one of these non-classical cases, that is in the class of *many-valued* Gödel justification logics. As introduced by Ghari [7] and Pischke [24], these arise by replacing the classical boolean base of justification logics with infinite (that is $[0,1]$-valued) propositional Gödel logic. Gödel logic itself occupies an interesting position in both the realm of many-valued logics and of the intermediate logics, that is propositional logics in between intuitionistic and classical logic with respect to expressive strength. For one, it is characterized as a t-norm based $[0,1]$-valued logics in the sense of Hájek [12], representing the case of the minimum t-norm. In this framework, it is e.g. the only instance that fulfils the classical deduction theorem. For another, the various Gödel logics (accounting for the finite and the infinite-valued versions) are particularly well-behaved intermediate logics, originating from Gödel's investigations into the finite-valued cases in [9]. The infinite-valued version was developed by Dummett [5].

In terms of its justification extensions, Gödel logic marks an interesting case of an intermediate justification logic, as it arises as a natural generalization of the classical case to the base of Gödel logic, both in a proof-theoretic way as well as in a model-theoretic way. However, as show in [25], it does not satisfy a natural analogue of the classical realization theorem with respect to the standard Gödel modal logics as introduced by Caicedo and Rodriguez in [3, 4].

Semantically, similar to the propositional case of Gödel logic, the situation for satisfiability and validity is now more diverse in the many-valued justification logics. Instead of studying basic sets of semantically satisfiable and valid formulas, we study 1-valued and positive satisfiability and validity with respect to the $[0,1]$-valued

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analogue of the Kripke-Fitting semantics for classical justification logic. In the propositional case, these new notions of satisfiability and validity are however resolved by simple reductions to the classical cases, as presented in Hájek's work [12].

We lift this method to the cases of Gödel justification logic and reduce the corresponding decidability problems to the classical cases. That this is possible, like in the non-modal propositional case, is centered around the observation that in Mkrtychev-semantics the modal formulas $t : \phi$ behave like additional atomic formulae next to the propositional variables p_i. Mkrtychev himself called formulas of the type $t : \phi$ "pseudo-atomic" in [23]. In contrast, this semantical method would not immediately apply for the standard Gödel modal logics, as the interpretation of the $\Box$-modality over Gödel-Kripke models is linked to the interpretation of its input formula.

2. A review of justification logics

We define the set of justification terms Jt as

$$Jt : t ::= c \mid x \mid [t+t] \mid [t \cdot t] \mid !t \mid ?t$$

where $c \in C := \{c_i \mid i \in \mathbb{N}\}$ is a justification constant and $x \in V := \{x_i \mid i \in \mathbb{N}\}$ is a justification variable. The corresponding language of justification logics $\mathcal{L}_J$ is then given as

$$\mathcal{L}_J : \phi ::= \bot \mid \top \mid p \mid (\phi \to \phi) \mid (\phi \wedge \phi) \mid t : \phi$$

with $t \in Jt$ and $p \in Var := \{p_i \mid i \in \mathbb{N}\}$ a propositional variable. We introduce the binary connective $\vee$ as the following abbreviation:

$$\phi \vee \psi := ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi).$$

We define the functions var and sub as follows:

$$\begin{aligned} &\mathrm{var}(\bot) = \mathrm{var}(\top) = \emptyset, && \mathrm{sub}(\bot) = \{\bot\}, \mathrm{sub}(\top) = \{\top\}, \\ &\mathrm{var}(p) = \{p\}, && \mathrm{sub}(p) = \{p\}, \\ &\mathrm{var}(\phi \wedge \psi) = \mathrm{var}(\phi) \cup \mathrm{var}(\psi), && \mathrm{sub}(\phi \wedge \psi) = \mathrm{sub}(\phi) \cup \mathrm{sub}(\psi) \cup \{\phi \wedge \psi\}, \\ &\mathrm{var}(\phi \to \psi) = \mathrm{var}(\phi) \cup \mathrm{var}(\psi), && \mathrm{sub}(\phi \to \psi) = \mathrm{sub}(\phi) \cup \mathrm{sub}(\psi) \cup \{\phi \to \psi\}, \\ &\mathrm{var}(t : \phi) = \mathrm{var}(\phi), && \mathrm{sub}(t : \phi) = \mathrm{sub}(\phi) \cup \{t : \phi\}. \end{aligned}$$

2.1. **The classical case.** For describing the proof systems for the various common justification logics, we fix an axiomatization for classical propositional logic. This is in general of no greater importance but will have a certain influence on the concept of *constant specification* later. Thus, in this paper, we consider the following proof theoretic access:

Definition 2.1. The calculus for propositional Gödel logic, denoted $\mathcal{G}$, is given by the following axiom schemes and rules:

($A1$): $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$
($A2$): $(\phi \wedge \psi) \to \phi$
($A3$): $(\phi \wedge \psi) \to (\psi \wedge \phi)$
($A5a$): $(\phi \to (\psi \to \chi)) \to ((\phi \wedge \psi) \to \chi)$
($A5b$): $((\phi \wedge \psi) \to \chi) \to (\phi \to (\psi \to \chi))$
($A6$): $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$
($A7$): $\bot \to \phi$
($G4$): $\phi \to (\phi \wedge \phi)$
($\top$): $\top \leftrightarrow \neg\bot$
(MP): From ϕ and $\phi \to \psi$, infer ψ.

We define the calculus $\mathcal{CL}$ as the extension of $\mathcal{G}$ by the scheme $\phi \vee \neg\phi$.

Theorem 2.2 (Hájek [12])**.** *Classical propositional logic is axiomatized by* $\mathcal{CL}$.

Definition 2.3. We define the following proof systems over $\mathcal{L}_J$:

(1) We define the calculus $\mathcal{J}_0$ as follows:
  (CL): The axiom schemes of the calculus $\mathcal{CL}$.
  (J): $t : (\phi \to \psi) \to (s : \phi \to [t \cdot s] : \psi)$,
  ($+$): $t : \phi \to [t+s] : \phi$, $s : \phi \to [t+s] : \phi$,
  (MP): From $\phi \to \psi$ and ϕ, infer ψ.
(2) $\mathcal{JT}_0$ is defined as $\mathcal{J}$ with $(F) : t : \phi \to \phi$,
(3) $\mathcal{J}4_0$ is defined as $\mathcal{J}$ with $(PI) : t : \phi \to !t : t : \phi$,

(4) $\mathcal{LP}_0$ is defined as $\mathcal{J}4$ with (F),
(5) $\mathcal{J}45_0$ is defined as $\mathcal{J}4$ with $(NI) : \neg t : \phi \to ?t : \neg t : \phi$,
(6) $\mathcal{JT}45_0$ is defined as $\mathcal{J}45$ with (F).

For any of the proof systems $\mathcal{S}$, we write $Th_{\mathcal{S}} := \{\phi \in \mathcal{L}_J \mid \vdash_{\mathcal{S}} \phi\}$. A constant specification for a such a proof system $\mathcal{S}$ over $\mathcal{L}_J$ is a set CS of formulas of the form

$$c_{i_n} : c_{i_{n-1}} : \cdots : c_{i_1} : \phi$$

where $n \geq 1$, $c_{i_k} \in C$ for all k and ϕ is an axiom instance of $\mathcal{S}$. Also, we expect CS to be downwards closed.

We call CS axiomatically appropriate for $\mathcal{S}$, if

(1) for every axiom instance ϕ for $\mathcal{S}$, there is a constant $c \in C$ s.t. $c : \phi \in CS$,
(2) if $c : \phi \in CS$, then $d : c : \phi \in CS$ for some constant $d \in C$.

For a given constant specification CS for a system $\mathcal{S}$, we denote by $\mathcal{S}_{CS}$ the system $\mathcal{S}$ extended by the necessitation rule:

$$\text{From } c : \phi \in CS, \text{ infer } c : \phi.$$

Definition 2.4. A Mkrtychev model is a structure $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ where

(1) $\mathcal{E} : Jt \times \mathcal{L}_J \to \{0, 1\}$,
(2) $e : Var \to \{0, 1\}$,

and which satisfies

(i) $\mathcal{E}(t, \phi \to \psi) = 1$ and $\mathcal{E}(s, \phi) = 1$ implies $\mathcal{E}(t \cdot s, \psi) = 1$ for all $t, s \in Jt$, $\phi, \psi \in \mathcal{L}_J$,
(ii) $\mathcal{E}(t, \phi) = 1$ or $\mathcal{E}(s, \phi) = 1$ implies $\mathcal{E}(t + s, \phi) = 1$ for all $t, s \in Jt$, $\phi \in \mathcal{L}_J$.

The class of all Mkrtychev models is denoted by M. We define the relation $\models$ between M-models $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ and the language $\mathcal{L}_J$ as follows:

- $\mathfrak{M} \not\models \bot$, $\mathfrak{M} \models \top$,
- $\mathfrak{M} \models p$ iff $e(p) = 1$ for $p \in Var$,
- $\mathfrak{M} \models (\phi \to \psi)$ iff $\mathfrak{M} \models \phi$ implies $\mathfrak{M} \models \psi$,
- $\mathfrak{M} \models (\phi \wedge \psi)$ iff $\mathfrak{M} \models \phi$ and $\mathfrak{M} \models \psi$,
- $\mathfrak{M} \models t : \phi$ iff $\mathcal{E}(t, \phi) = 1$.

For a set $\Gamma \subseteq \mathcal{L}_J$, we write $\mathfrak{M} \models \Gamma$ if $\forall \phi \in \Gamma : \mathfrak{M} \models \phi$.

Definition 2.5. A Mkrtychev model $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ is called a

(1) MT-model if $\mathcal{E}(t, \phi) = 1$ implies $\mathfrak{M} \models \phi$ for all $t \in Jt$, $\phi \in \mathcal{L}_J$,
(2) $\mathsf{M4}$-model if $\mathcal{E}(t, \phi) = 1$ implies $\mathcal{E}(!t, t : \phi) = 1$ for all $t \in Jt$, $\phi \in \mathcal{L}_J$,
(3) MLP-model if (1) and (2),
(4) $\mathsf{M45}$-model if (2) and $\mathcal{E}(t, \phi) = 0$ implies $\mathcal{E}(?t, \neg t : \phi) = 1$ for all $t \in Jt$, $\phi \in \mathcal{L}_J$,
(5) $\mathsf{MT45}$-model if (1) and (4).

Definition 2.6. We say that a Mkrtychev model $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ respects a constant specification CS (for some proof system $\mathcal{S}$) if $\mathcal{E}(c, \phi) = 1$ for any $c : \phi \in CS$. For a class of Mkrtychev models C we denote, by C_{CS}, the class of all Mkrtychev models in C which respect CS.

Definition 2.7. Let C be a class of Mkrtychev models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We say that ϕ is a consequence of Γ in C, written $\Gamma \models^M_{\mathsf{C}} \phi$, if for any $\mathfrak{M} \in \mathsf{C}$, if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \phi$.

Theorem 2.8 (Completeness; Mkrtychev [23], Kuznets [16], Studer [26])**.** *Let $\mathcal{JL}_0 \in \{\mathcal{J}_0, \mathcal{JT}_0, \mathcal{J}4_0, \mathcal{LP}_0, \mathcal{J}45_0, \mathcal{JT}45_0\}$ and CS be a constant specification for $\mathcal{JL}_0$. Let $\mathsf{MJL} \in \{\mathsf{M}, \mathsf{MT}, \mathsf{M4}, \mathsf{MLP}, \mathsf{M45}, \mathsf{MT45}\}$ be the corresponding class of M-models to $\mathcal{JL}_0$. Then for all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$:*

$$\Gamma \vdash_{\mathcal{JL}_{CS}} \phi \text{ iff } \Gamma \models_{\mathsf{MJL}_{\mathsf{CS}}} \phi.$$

Historically, Mkrtychev semantics were developed as a semantics for the logic of proofs $\mathcal{LP}$ not based on arithmetic provability by Mkrtychev [23]. The adaptation of the semantics to the logics $\mathcal{J}, \mathcal{JT}, \mathcal{J}4$ is due to Kuznets [16]. The completeness theorem regarding Mkrtychev models for $\mathcal{J}45$ and $\mathcal{JT}45$ appear, in an adapted form, in [26]. In fact, in the literature there is a slightly different semantics based Mkrtychev-models, called pre-models by Mkrtychev in [23]. These have a relevance for justification logics which include the factivity axiom scheme, excluding versions which negative introspection. For the classical case, this is also discussed in [26], where Studer actually writes about *strong pre-models*, which are the Mkrtychev-models here. The many-valued (Gödel) case of Mkrtychev's pre-model semantics for Gödel justification logics was considered in [25] in the context of realizability.

Definition 2.9. Let C be a class of M-models. We define

(1) $\text{SAT}^{\mathsf{C}} := \{\phi \in \mathcal{L}_J \mid \exists \mathfrak{M} \in \mathsf{C} : \mathfrak{M} \models \phi\}$,
(2) $\text{TAUT}^{\mathsf{C}} := \{\phi \in \mathcal{L}_J \mid \forall \mathfrak{M} \in \mathsf{C} : \mathfrak{M} \models \phi\}$.

We have the following decidability and complexity results for the set of theorems:

Theorem 2.10 (Kuznets [17], Studer [26]). *Let*

(1) $\mathcal{JL}_{CS} \in \{\mathcal{J}_{CS}, \mathcal{JT}_{CS}, \mathcal{J}4_{CS}, \mathcal{LP}_{CS}\}$ *for decidable almost schematic* CS*, or*
(2) $\mathcal{JL}_{CS} \in \{\mathcal{J}45_{CS}, \mathcal{JT}45_{CS}\}$ *for finite* CS*.*

Then $Th_{\mathcal{JL}_{CS}}$ *is decidable.*

The above theorem collects some of the current results in decidability and complexity of justification logics. More precisely, part (1) as formulated above is due to Kuznets [17] while part (2) is due to Studer [26].

As shown in Kuznets' work [17], the condition that the constant specification is almost schematic can not be dropped. It is, however, not known if it can be weakened. The question if $\mathcal{J}45_{CS}, \mathcal{JT}45_{CS}$ is decidable with a schematic CS is also still open.

In terms of complexity, there is a trivial lower bound based on the fact that the justification logics are conservative extensions over classical propositional logic.

Theorem 2.11. *Let* $\mathcal{JL}_0 \in \{\mathcal{J}_0, \mathcal{JT}_0, \mathcal{J}4_0, \mathcal{LP}_0, \mathcal{J}45_0, \mathcal{JT}45_0\}$ *and* CS *be a constant specification for* $\mathcal{JL}_0$*. Then* $Th_{\mathcal{JL}_{CS}}$ *is* co-NP*-hard.*

This theorem is basically folklore, mentioned in passing already in [16]. Note that not all instances of $Th_{\mathcal{JL}_{CS}}$ are decidable in the above theorem. In terms of more tight complexity estimates, we have the following theorem.

Theorem 2.12 (Milnikel [21]). *For* $\mathcal{JL}_{CS} \in \{\mathcal{J}_{CS}, \mathcal{JT}_{CS}, \mathcal{J}4_{CS}, \mathcal{LP}_{CS}\}$ *with a decidable almost schematic* CS*,* $Th_{\mathcal{JL}_{CS}}$ *is in* Π_2^p*. Further, we have:*

(1) If $\mathcal{JL}_{CS}$ *is either*
   (a) $\mathcal{J}4_{CS}$ *with a decidable schematic* CS*, or*
   (b) $\mathcal{LP}_{CS}$ *with a decidable schematically injective axiomatically appropriate* CS*,*
   then $Th_{\mathcal{JL}_{CS}}$ *is* Π_2^p*-complete.*
(2) If $\mathcal{JL}_{CS} = \mathcal{LP}_{CS}$ *with a decidable injective* CS*, then* $Th_{\mathcal{JL}_{CS}}$ *is* co-NP*-complete.*

The above results then may be easily be rephrased in terms of SAT^{C} and TAUT^{C}, using $\text{TAUT}^{\mathsf{MJL_{cs}}} = Th_{\mathcal{JL}_{CS}}$ and $\phi \in \text{TAUT}^{\mathsf{C}}$ iff $\neg\phi \in \text{SAT}^{\mathsf{C}}$.

2.2. The non-classical case.

We motivate the Gödel justification logics in a proof theoretic way. For this, we define the following proof systems over $\mathcal{L}_J$. For $\mathcal{S} \in \{\mathcal{J}_0, \mathcal{JT}_0, \mathcal{J}4_0, \mathcal{LP}_0, \mathcal{J}45_0, \mathcal{JT}45_0\}$, we define $\mathcal{GS}$ to be the reduct of $\mathcal{S}$ without the scheme $\phi \vee \neg\phi$.

Definition 2.13. A Gödel-Mkrtychev model is a structure $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ where

(1) $\mathcal{E} : Jt \times \mathcal{L}_J \to [0, 1]$,
(2) $e : Var \to [0, 1]$,

and which satisfies

(i) $\mathcal{E}(t, \phi \to \psi) \odot \mathcal{E}(s, \phi) \leq \mathcal{E}(t \cdot s, \psi)$ for all $t, s \in Jt$, $\phi, \psi \in \mathcal{L}_J$,
(ii) $\mathcal{E}(t, \phi) \oplus \mathcal{E}(s, \phi) \leq \mathcal{E}(t + s, \phi)$ for all $t, s \in Jt$, $\phi \in \mathcal{L}_J$.

We denote the class of all Gödel-Mkrtychev models by GM. We call a GM-model $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ *crisp* if both $\mathcal{E}$ and e only take values in $\{0, 1\}$. For a class of GM-models C, we denote its subclass of crisp models by C^{c}.

For a GM-model $\mathfrak{M} = \langle \mathcal{E}, e\rangle$, we define its evaluation function $|\cdot|_{\mathfrak{M}} : \mathcal{L}_J \to [0, 1]$ as follows:

- $|\bot|_{\mathfrak{M}} = 0$, $|\top|_{\mathfrak{M}} = 1$,
- $|p|_{\mathfrak{M}} = e(p)$ for $p \in Var$,
- $|\phi \to \psi|_{\mathfrak{M}} = |\phi|_{\mathfrak{M}} \Rightarrow |\psi|_{\mathfrak{M}}$,
- $|\phi \wedge \psi|_{\mathfrak{M}} = |\phi|_{\mathfrak{M}} \odot |\psi|_{\mathfrak{M}}$,
- $|t : \phi|_{\mathfrak{M}} = \mathcal{E}(t, \phi)$.

We may extend this evaluation to sets of formulas $\Gamma \subseteq \mathcal{L}_J$ by setting $|\Gamma|_{\mathfrak{M}} = \inf_{\phi\in\Gamma}\{|\phi|_{\mathfrak{M}}\}$. We write $\mathfrak{M} \models \phi$ if $|\phi|_{\mathfrak{M}} = 1$ and $\mathfrak{M} \models \Gamma$ if $\mathfrak{M} \models \phi$ for all $\phi \in \Gamma$. Note, that $\mathfrak{M} \models \Gamma$ is equivalent with $|\Gamma|_{\mathfrak{M}} = 1$.

Definition 2.14. A GM-model $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ is called a

(1) GMT-model if $\mathcal{E}(t, \phi) \leq |\phi|_{\mathfrak{M}}$ for all $t \in Jt$, $\phi \in \mathcal{L}_J$,
(2) $\mathsf{GM4}$-model if $\mathcal{E}(t, \phi) \leq \mathcal{E}(!t, t : \phi)$ for all $t \in Jt$, $\phi \in \mathcal{L}_J$,
(3) GMLP-model if (1) and (2),

(4) $\mathsf{GM45}$-model if (2) and $\sim\mathcal{E}(t,\phi) \leq \mathcal{E}(?t, \neg t:\phi) = 1$ for all $t \in Jt$, $\phi \in \mathcal{L}_J$,
(5) $\mathsf{GMT45}$-model if (1) and (4).

Definition 2.15. Let C be class of GM-models. For $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$, we say that Γ 1-entails ϕ in C, written $\Gamma \models_{\mathsf{C}} \phi$, if for any models $\mathfrak{M} \in \mathsf{C}$, if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \phi$.

The main theorem on Gödel justification logics used in this paper is the completeness theorem for the above systems and the Gödel-Mkrtychev models introduced before.

Theorem 2.16 (Completeness, P. [24])**.** *Let $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$ and CS be a constant specification for $\mathcal{GJL}_0$. Let $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$ be the corresponding class of Gödel-Mkrtychev models for $\mathcal{GJL}_0$. Then for all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$:*

$$\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi \text{ iff } \Gamma \models_{\mathsf{GMJL_{CS}}} \phi.$$

In Gödel logic, there is another common notion of entailment which can be transferred to Gödel justification logics.

Definition 2.17. Let C be a class of GM-models and $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$. We say that Γ entails ϕ in C, written $\Gamma \models_{\mathsf{C}\leq} \phi$, if for any $\mathfrak{M} \in \mathsf{C}$, $|\Gamma|_{\mathfrak{M}} \leq |\phi|_{\mathfrak{M}}$.

Lemma 2.18 (P. [24])**.** *Let $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$, and CS be a constant specification for the corresponding proof system $\mathcal{GJL}_0$. Then $\Gamma \models_{\mathsf{GMJL_{CS}}} \phi$ iff $\Gamma \models_{\mathsf{GMJL_{CS}}\leq} \phi$ for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_J$.*

As it is easily observed, the classical Mkrtychev models are exactly the crisp Gödel-Mkrtychev models of their respective class, that is for a constant specification CS[1],

$$\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$$

and the corresponding class of classical Mkrtychev models

$$\mathsf{MJL} \in \{\mathsf{M}, \mathsf{MT}, \mathsf{M4}, \mathsf{MLP}, \mathsf{M45}, \mathsf{MT45}\},$$

it holds that

$$\mathsf{GMJL}^{\mathsf{c}}_{\mathsf{CS}} = \mathsf{MJL}_{\mathsf{CS}}.$$

Further, as $\odot, \Rightarrow$ are natural generalizations of the classical truth functions for $\wedge, \rightarrow$, for any model $\mathfrak{M} \in \mathsf{M}$ it holds that

$$\mathfrak{M} \models \phi \text{ iff } |\phi|_{\mathfrak{M}} = 1$$

and

$$\mathfrak{M} \not\models \phi \text{ iff } |\phi|_{\mathfrak{M}} = 0.$$

Through this identification, we will also apply the notation concerning GM-models to M-models.

Now, in the context of many-valued Mkrtychev models, there are various graded notions of satisfiability and validity to be considered.

Definition 2.19. Let C be a class of GM-models. We define
(1) $\mathrm{SAT}^{\mathsf{C}}_1 := \{\phi \in \mathcal{L}_J \mid \exists \mathfrak{M} \in \mathsf{C} : |\phi|_{\mathfrak{M}} = 1\}$,
(2) $\mathrm{SAT}^{\mathsf{C}}_{pos} := \{\phi \in \mathcal{L}_J \mid \exists \mathfrak{M} \in \mathsf{C} : |\phi|_{\mathfrak{M}} > 0\}$,
(3) $\mathrm{TAUT}^{\mathsf{C}}_1 := \{\phi \in \mathcal{L}_J \mid \forall \mathfrak{M} \in \mathsf{C} : |\phi|_{\mathfrak{M}} = 1\}$,
(4) $\mathrm{TAUT}^{\mathsf{C}}_{pos} := \{\phi \in \mathcal{L}_J \mid \forall \mathfrak{M} \in \mathsf{C} : |\phi|_{\mathfrak{M}} > 0\}$.

3. Positive and 1-satisfiability

To study satisfiability in this section and validity later, we translate the justification assertions into an augmented propositional language. This translation is along the lines of the investigations of Caicedo and Rodriguez in [3, 4] for Gödel modal logics and was adapted in [24] to the case of Gödel justification logics to study completeness properties.

For this, we fix a propositional language $\mathcal{L}_0(X)$ over a countably infinite set of variables X by the BNF

$$\mathcal{L}_0(X) : \phi ::= \bot \mid \top \mid p \mid (\phi \wedge \phi) \mid (\phi \rightarrow \phi).$$

This now yields the ground for the following definition.

Definition 3.1. We set $Var^\star := Var \cup \{\phi_t \mid \phi \in \mathcal{L}_J, t \in Jt\}$ and $\mathcal{L}_0^\star := \mathcal{L}_0(Var^\star)$. We define $\star : \mathcal{L}_J \to \mathcal{L}_0^\star$ as
- $\bot \mapsto \bot$, $\top \mapsto \top$,

[1] We use this formulation throughout the paper whenever the ambient system for which this set is a constant specification for is not important.

- $p \mapsto p$ for $p \in Var$,
- $(\phi \wedge \psi) \mapsto (\phi^\star \wedge \psi^\star)$,
- $(\phi \to \psi) \mapsto (\phi^\star \to \psi^\star)$,
- $t : \phi \mapsto \phi_t$.

We may extend this to sets of formulas $\Gamma \subseteq \mathcal{L}_J$ by $\Gamma^\star := \{\phi^\star \mid \phi \in \Gamma\}$. It is straightforward to show that $\star$ is bijective between $\mathcal{L}_J$ and $\mathcal{L}_0^\star$. We denote this inverse with $\star^{-1}$.

We now define the central concept of this section, so called regulators, which are derived from the decidability investigations in [12]. A regulator is a collection of atomic and pseudo-atomic variables (in their $\star$-translated form), which are to be regarded as *null*.

Definition 3.2. We call $I \subseteq Var^\star$ a *regulator*. Further, we set $I_p := I \cap Var$ and $I_j := I \cap (Var^\star \setminus Var)$.

As regulators specify which atomic propositions and pseudo-atomic propositions are regarded as null, with every regulator $I \subseteq Var^\star$, we may associate a canonical map $\cdot^I : \mathcal{L}_J \to \mathcal{L}_J$ which transforms every formula to a shorter version by redacting it based on the information in the regulator and the semantical interpretations of the propositional connectives.

- $\bot \mapsto \bot$, $\top \mapsto \top$,
- $p \mapsto \begin{cases} \bot, & \text{if } p \in I_p \\ p, & \text{otherwise} \end{cases}$ for $p \in Var$,
- $\phi \wedge \psi \mapsto \begin{cases} \bot, & \text{if } \phi^I = \bot \text{ or } \psi^I = \bot \\ \phi^I \wedge \psi^I, & \text{otherwise} \end{cases}$,
- $\phi \to \psi \mapsto \begin{cases} \top, & \text{if } \phi^I = \bot \\ \bot, & \text{if } \psi^I = \bot \text{ and } \phi^I \neq \bot, \\ \phi^I \to \psi^I, & \text{otherwise} \end{cases}$
- $t : \phi \mapsto \begin{cases} \bot, & \text{if } \phi_t \in I_j \\ t : \phi, & \text{otherwise} \end{cases}$.

The following definition now gives conditions for regulators to be compliant with the various Gödel-Mkrtychev model classes of the semantics for Gödel justification logics. As regulators are intended to specify 0-valued atomic and pseudo-atomic variables, they have to comply with the various *closure* conditions the the evidence function $\mathcal{E}$ of a Gödel-Mkrtychev model $\langle \mathcal{E}, e\rangle$ if they want to accurately model the 0-level of this function $\mathcal{E}$.

Definition 3.3. Let I be a regulator.

(a) For a constant specification CS, I is called a
  (1) $\mathsf{GM}_{\mathsf{CS}}$-*regulator* if:
    (i) When $\psi_{[t\cdot s]} \in I_j$, then $\forall \phi \in \mathcal{L}_J : \phi_s \in I_j$ or $(\phi \to \psi)_t \in I_j$.
    (ii) When $\phi_{[t+s]} \in I_j$, then $\phi_t, \phi_s \in I_j$.
    (iii) When $c : \phi \in CS$, then $\phi_c \notin I_j$.
  (2) $\mathsf{GM4}_{\mathsf{CS}}$-*regulator* if additionally to (1):
    (i) When $(t : \phi)_{!t} \in I_j$, then $\phi_t \in I_j$.
  (4) $\mathsf{GM45}_{\mathsf{CS}}$-*regulator* if additionally to (2):
    (i) When $(\neg t : \phi)_{?t} \in I_j$, then $\phi_t \notin I_j$.
  (5) $\mathsf{GMT}_{\mathsf{CS}}$-*regulator* if additionally to (1):
    (i) When $\phi^I = \bot$, then $\phi_t \in I_j$ for all $t \in Jt$.
  (6) $\mathsf{GMLP}_{\mathsf{CS}}$-*regulator* if (2) and (4).
  (7) $\mathsf{GMT45}_{\mathsf{CS}}$-*regulator* if (3) and (4).

(b) For a GM-model $\mathfrak{M} = \langle \mathcal{E}, e\rangle$, $\mathfrak{M}$ is called I-regular if
  (a) $e(p) = 0$ iff $p \in I_p$ for $p \in Var$,
  (b) $\mathcal{E}(t, \phi) = 0$ iff $\phi_t \in I_j$ for $\phi \in \mathcal{L}_J$, $t \in Jt$.

The following lemmas establish basic facts about the workings of regulators and their corresponding canonical map.

Lemma 3.4. *Let I be a regulator. For all $\phi \in \mathcal{L}_J$, either $\phi^I = \bot$ or $\bot \notin \mathrm{sub}((\phi^I)^\star)$.*

Proof. Let $I \subseteq Var^\star$ be an arbitrary regulator. First, we note that the cases are mutually exclusive. For this, note that if $\phi^I = \bot$, then $(\phi^I)^\star = \bot$ and thus $\bot \in \mathrm{sub}((\phi^I)^\star) = \{\bot\}$. We now do an induction on $\mathcal{L}_J$.

(IB): For the case of $\bot$, we always have $\phi^I = \bot$. For the case of $\top$, we have $\bot \notin \{\top\} = \mathrm{sub}(\top)$. For the case of $p \in Var$, if $p \notin I_p$, then $p^I = p$ and thus $\bot \notin \{p\} = \mathrm{sub}(p) = \mathrm{sub}((p^I)^\star)$

(IS): Let the claim hold for ψ, χ arbitrary formulas. We divide between the following cases:

(1) For $\psi \wedge \chi$, if $(\psi \wedge \chi)^I \neq \bot$, then $(\psi \wedge \chi)^I = \psi^I \wedge \chi^I$ and $\psi^I, \chi^I \neq \bot$. By **(IH)**, we have
$$\bot \notin \mathrm{sub}((\psi^I)^\star) \cup \mathrm{sub}((\chi^I)^\star).$$
Now, $((\psi \wedge \chi)^I)^\star = (\psi^I \wedge \chi^I)^\star = (\psi^I)^\star \wedge (\chi^I)^\star$. Thus
$$\begin{aligned}\mathrm{sub}(((\psi \wedge \chi)^I)^\star) &= \mathrm{sub}((\psi^I)^\star \wedge (\chi^I)^\star)\\ &= \mathrm{sub}((\psi^I)^\star) \cup \mathrm{sub}((\chi^I)^\star) \cup \{(\psi^I)^\star \wedge (\chi^I)^\star\}\end{aligned}$$
where $\bot \neq (\psi^I)^\star \wedge (\chi^I)^\star$.

(2) For $\psi \to \chi$, if $(\psi \to \chi)^I \neq \bot$, either $(\psi \to \chi)^I = \top$ or $(\psi \to \chi)^I = \psi^I \to \chi^I$. For the former, $\bot \notin \{\top\} = \mathrm{sub}(((\psi \to \chi)^I)^\star)$. For the latter, we may proceed as in case (1).

(3) In the case of $t : \psi$, if $(t : \psi)^I \neq \bot$, then $(t : \psi)^I = t : \psi$ and thus $\mathrm{sub}(((t : \psi)^I)^\star) = \mathrm{sub}((t : \psi)^\star) = \{\psi_t\}$ and $\bot \neq \psi_t$.

□

Lemma 3.5. *Let I be a regulator and $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ be an I-regular* GM*-model. Then $|\phi^I|_{\mathfrak{M}} = 0$ iff $\phi^I = \bot$ for all $\phi \in \mathcal{L}_J$.*

Proof. Let I be an arbitrary regulator and $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ an arbitrary I-regular GM-model. Now, for all $\phi \in \mathcal{L}_J$, if $\phi^I = \bot$, then $|\phi^I|_{\mathfrak{M}} = |\bot|_{\mathfrak{M}} = 0$ by definition. We prove
$$\forall \phi \in \mathcal{L}_J : |\phi^I|_{\mathfrak{M}} = 0 \text{ implies } \phi^I = \bot$$
by induction on the structure of $\mathcal{L}_J$.

(IB): For the case of $\bot$, we have $\bot^I = \bot$, which immediately validates the claim. For the case of $\top$, we always have $|\top^I|_{\mathfrak{M}} = |\top|_{\mathfrak{M}} = 1 > 0$. Now, in the case of $p \in Var$, we have either $p \in I_p$ or $p \notin I_p$. For the former, we have $p^I = \bot$. For the latter, we have $p^I = p \neq \bot$ but as $\mathfrak{M}$ is I-regular, we have $|p|_{\mathfrak{M}} > 0$.

(IS): Let ψ, χ be two formulas for which the claim holds. We divide between the following cases:

(1) For $\psi \to \chi$, we may have $(\psi \to \chi)^I = \top$, i.e. $|(\psi \to \chi)^I|_{\mathfrak{M}} = 1$. We may alternatively have $(\psi \to \chi)^I = \bot$, where we are done.
Lastly, we may have $(\psi \to \chi)^I = \psi^I \to \chi^I$, i.e. $\psi^I \neq \bot$ and $\chi^I \neq \bot$. By **(IH)**, we have thus $|\psi^I|_{\mathfrak{M}}, |\chi^I|_{\mathfrak{M}} > 0$ and thus $|(\psi \to \chi)^I|_{\mathfrak{M}} = |\psi^I|_{\mathfrak{M}} \Rightarrow |\chi^I|_{\mathfrak{M}} \geq |\chi^I|_{\mathfrak{M}} > 0$.

(2) For $\psi \wedge \chi$, we may have $(\psi \wedge \chi)^I = \bot$ or $(\psi \wedge \chi)^I = \psi^I \wedge \chi^I$. For the latter, we have again $\psi^I \neq \bot$ and $\chi^I \neq \bot$ and by **(IH)** $|\psi^I|_{\mathfrak{M}}, |\chi^I|_{\mathfrak{M}} > 0$. Thus $|(\psi \wedge \chi)^I| = |\psi^I|_{\mathfrak{M}} \odot |\chi^I|_{\mathfrak{M}} > 0$.

(3) For $t : \psi$, we have either $(t : \psi)^I = \bot$ or $(t : \psi)^I = t : \psi$. For the latter, $\psi_t \notin I_j$. As $\mathfrak{M}$ is I-regular, we have $|t : \psi|_{\mathfrak{M}} = \mathcal{E}(t, \psi) > 0$.

□

Lemma 3.6. *Let I be a regulator and $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ be an I-regular* GM*-model. Then $|\phi|_{\mathfrak{M}} = |\phi^I|_{\mathfrak{M}}$ for all $\phi \in \mathcal{L}_J$.*

Proof. Let I be an arbitrary regulator and $\mathfrak{M} = \langle \mathcal{E}, e\rangle$ an arbitrary I-regular GM-model. We prove the claim by induction on the structure of $\mathcal{L}_J$.

(IB): $\cdot^I$ is the identity for $\bot, \top$. In the case of $p \in Var$, we either have $p \in I_p$, i.e. $p^I = \bot$ and as $\mathfrak{M}$ is I-regular, we have $|p|_{\mathfrak{M}} = e(p) = 0 = |\bot|_{\mathfrak{M}} = |p^I|_{\mathfrak{M}}$. For $p \notin I_p$, we have that $p^I = p$, where the claim follows naturally.

(IS): Suppose $\psi, \chi \in \mathcal{L}_J$ are formulas for which the claim holds. We divide between the following cases:

(1) Considering $\psi \to \chi$, we maybe have $(\psi \to \chi)^I = \top$, i.e. $\psi^I = \bot$. By **(IH)**, we have $|\psi|_{\mathfrak{M}} = |\psi^I|_{\mathfrak{M}} = 0$ and thus $|\psi \to \chi|_{\mathfrak{M}} = |\psi|_{\mathfrak{M}} \Rightarrow |\chi|_{\mathfrak{M}} = 1$.
We may also have $(\psi \to \chi)^I = \bot$, i.e. $\chi^I = \bot$ but $\psi^I \neq \bot$, thus again by **(IH)**, we have $|\chi|_{\mathfrak{M}} = 0$ and by Lem. 3.5, as $\psi^I \neq \bot$, we have $|\psi|_{\mathfrak{M}} = |\psi^I|_{\mathfrak{M}} > 0$. Thus, $|\psi \to \chi|_{\mathfrak{M}} = |\psi|_{\mathfrak{M}} \Rightarrow |\chi|_{\mathfrak{M}} = 0$.
Lastly, we may have $(\psi \to \chi)^I = \psi^I \to \chi^I$. Thus, by **(IH)**, we have $|(\psi \to \chi)^I|_{\mathfrak{M}} = |\psi^I|_{\mathfrak{M}} \Rightarrow |\chi^I|_{\mathfrak{M}} = |\psi|_{\mathfrak{M}} \Rightarrow |\chi|_{\mathfrak{M}} = |\psi \to \chi|_{\mathfrak{M}}$.

(2) For $\psi \wedge \chi$, we may have $(\psi \wedge \chi)^I = \bot$, i.e. $\psi^I = \bot$ or $\chi^I = \bot$. Thus, by **(IH)**, $|\psi|_{\mathfrak{M}} = 0$ or $|\chi|_{\mathfrak{M}} = 0$, i.e. $|\psi \wedge \chi|_{\mathfrak{M}} = 0$. On the other hand, we may have $(\psi \wedge \chi)^I = \psi^I \wedge \chi^I$. By **(IH)**, we have $|(\psi \wedge \chi)^I|_{\mathfrak{M}} = |\psi^I|_{\mathfrak{M}} \odot |\chi^I|_{\mathfrak{M}} = |\psi|_{\mathfrak{M}} \odot |\chi|_{\mathfrak{M}} = |\psi \wedge \chi|_{\mathfrak{M}}$.

(3) For $t : \psi$, either $(t : \psi)^I = \bot$, i.e. $\psi_t \in I_j$. Then $|t : \psi|_{\mathfrak{M}} = \mathcal{E}(t, \psi) = 0 = |(t : \psi)^I||_{\mathfrak{M}}$ as $\mathfrak{M}$ is I-regular. Otherwise, $(t : \psi)^I = t : \psi$, where the claim is immediate.

□

Corollary 3.7. Let I be a regulator and $\mathfrak{M}$ be an I-regular GM-model. For any $\phi \in \mathcal{L}_J$, if $\bot \notin \mathrm{sub}((\phi^I)^\star)$, then $|\phi|_{\mathfrak{M}} \neq 0$.

Proof. Let ϕ be s.t. $\bot \notin \text{sub}((\phi^I)^\star)$ for a regulator I. Let $\mathfrak{M}$ be an I-regular GM-model. By Lem. 3.4, we have that $\phi^I \neq \bot$. By Lem. 3.5, we have thus $|\phi^I|_{\mathfrak{M}} > 0$ and by Lem. 3.6, we thus have $|\phi|_{\mathfrak{M}} > 0$. □

Lemma 3.8. *Let* $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$ *and* $\mathsf{MJL} = \mathsf{GMJL}^{\mathsf{c}}$ *be the corresponding class of classical Mkrtychev models. Let* CS *be a constant specification. For all* $\phi \in \mathcal{L}_J$*, t.f.a.e.:*

(1) $\phi \in \text{SAT}_{pos}^{\mathsf{GMJL_{CS}}}$,
(2) there is a $\mathsf{GMJL_{CS}}$*-regulator* I *s.t.* $\phi^I \neq \bot$,
(3) $\phi \in \text{SAT}^{\mathsf{MJL_{CS}}}$.

Proof. Let ϕ be arbitrary. We do a round-robin proof.

(1) ⇒ (2): Suppose that $\phi \in \text{SAT}_{pos}^{\mathsf{GMJL_{CS}}}$, i.e. $\exists \mathfrak{M} \in \mathsf{GMJL_{CS}} : |\phi|_{\mathfrak{M}} > 0$. We define $I = I_p \cup I_j$ as follows:
(a) $I_p = \{p \in Var \mid |p|_{\mathfrak{M}} = 0\}$,
(b) $I_j = \{\psi_t \in Var^\star \setminus Var \mid \mathcal{E}(t, \psi) = 0\}$.
By definition, $\mathfrak{M}$ is I-regular and thus by Lem. 3.6, $|\phi^I|_{\mathfrak{M}} = |\phi|_{\mathfrak{M}} > 0$ and by Lem. 3.5, we have $\phi^I \neq \bot$. It remains to show that I is a $\mathsf{GMJL_{CS}}$-regulator. For this, we divide between the different cases for GMJL:
GM: We divide between the following cases:
(1) Suppose that $\psi_{[t\cdot s]} \in I_j$, thus $\mathcal{E}(t \cdot s, \psi) = 0$. As $\mathfrak{M}$ is a GM-model, we have for any $\phi \in \mathcal{L}_J$, that $\mathcal{E}(t, \phi \to \psi) \odot \mathcal{E}(s, \phi) = 0$, i.e. $\mathcal{E}(t, \phi \to \psi) = 0$ or $\mathcal{E}(s, \phi) = 0$. This gives $\phi_s \in I_j$ or $(\phi \to \psi)_t \in I_j$.
(2) Suppose $\phi_{[t+s]} \in I_j$, then $\mathcal{E}(t + s, \phi) = 0$ and thus $\mathcal{E}(t, \phi), \mathcal{E}(s, \phi) = 0$ as $\mathfrak{M}$ is a GM-model which gives $\phi_t, \phi_s \in I_j$.
(3) Let $c : \phi \in CS$, then as $\mathfrak{M}$ respects CS, we have $\mathcal{E}(c, \phi) = 1$ and thus $\phi_c \notin I_j$.
GM4: Suppose $(t : \phi)_{!t} \in I_j$. Thus $\mathcal{E}(!t, t : \phi) = 0$ and as $\mathfrak{M}$ is a $\mathsf{GM4}$-model, we have $\mathcal{E}(t, \phi) = 0$, i.e. $\phi_t \in I_j$. Otherwise we proceed as in the case for GM.
GM45: Suppose $(\neg t : \phi)_{?t} \in I_j$, i.e. $\mathcal{E}(?t, \neg t : \phi) = 0$ and as $\mathfrak{M}$ is a $\mathsf{GM45}$-model, we have $\sim \mathcal{E}(t, \phi) = 0$. Thus $\mathcal{E}(t, \phi) \neq 0$ and thus $\phi_t \notin I_j$.
GMT: Suppose $\phi^I = \bot$. Thus by Lem. 3.5 and Lem. 3.6 $|\phi|_{\mathfrak{M}} = |\phi^I|_{\mathfrak{M}} = 0$. As $\mathfrak{M}$ is a GMT-model, we have $\mathcal{E}(t, \phi) = 0$ for all $t \in Jt$. Thus $\phi_t \in I_j$ for all $t \in Jt$.
GMLP: This case is similar to the combination of the GMT and $\mathsf{GM4}$ cases.
GMT45: This case is similar to the combination of the GMT and $\mathsf{GM45}$ cases.

(2) ⇒ (3): Let $I \subseteq Var^\star$ be a $\mathsf{GMJL_{CS}}$-regulator s.t. $\phi^I \neq \bot$. We define $\mathfrak{M} = \langle \mathcal{E}, e \rangle$ s.t.
(a) $e(p) = \begin{cases} 0, & \text{if } p \in I_p \\ 1, & \text{otherwise} \end{cases}$ for $p \in Var$,
(b) $\mathcal{E}(t, \psi) = \begin{cases} 0, & \text{if } \psi_t \in I_j \\ 1, & \text{otherwise} \end{cases}$ for $\psi \in \mathcal{L}_J$, $t \in Jt$.
Note that $\mathfrak{M}$ is an I-regular model. We now show that $\mathfrak{M} \in \mathsf{MJL_{CS}}$. For this, we divide between the possible cases for $\mathsf{MJL}(\mathsf{GMJL}^{\mathsf{c}})$:
M: Suppose $\mathcal{E}(t \cdot s, \psi) \neq 1$, then $\psi_{[t\cdot s]} \in I_j$. Thus, for any $\phi \in \mathcal{L}_J$, we have $\phi_s \in I_j$ or $(\phi \to \psi)_t \in I_j$, i.e. $\mathcal{E}(s, \phi) = 0$ or $\mathcal{E}(t, \phi \to \psi) = 0$.

Suppose $\mathcal{E}(t + s, \phi) \neq 1$, then $\phi_{[t+s]} \in I_j$ and thus $\phi_t, \phi_s \in I_j$, i.e. $\mathcal{E}(t, \phi) = \mathcal{E}(s, \phi) = 0$.
M4: Suppose $\mathcal{E}(!t, t : \phi) \neq 0$, then $(t : \phi)_{!t} \in I_j$ and thus $\phi_t \in I_j$, i.e. $\mathcal{E}(t, \phi) = 0$. The rest follows as in the case for M.
M45: Suppose $\mathcal{E}(?t, \neg t : \phi) \neq 1$, i.e. $(\neg t : \phi)_{?t} \in I_j$, i.e. $\phi_t \notin I_j$, i.e. $\mathcal{E}(t, \phi) = 1$. The rest follows as in the case for $\mathsf{M4}$.
MT: Let $\mathcal{E}(t, \phi) = 1$, i.e. $\phi_t \notin I_j$. Thus $\phi^I \neq \bot$. By Lem. 3.5 and Lem. 3.6, $|\phi|_{\mathfrak{M}} = |\phi^I|_{\mathfrak{M}} \neq 0$. Thus $\mathfrak{M} \models \phi$.
MLP: This follows from the case for MT and $\mathsf{M4}$.
MT45: This follows as in the case for MT and $\mathsf{M45}$.
By Lem. 3.4, we have, as $\phi^I \neq \bot$, that $\bot \notin \text{sub}((\phi^I)^\star)$. Thus, $|\phi|_{\mathfrak{M}} \neq 0$ by Cor. 3.7, i.e. $|\phi|_{\mathfrak{M}} = 1$.

(3) ⇒ (1): Suppose $\phi \in \text{SAT}^{\mathsf{MJL_{CS}}}$, i.e. $\exists \mathfrak{M} \in \mathsf{MJL_{CS}} : \mathfrak{M} \models \phi$, i.e. $|\phi|_{\mathfrak{M}} = 1$. Naturally we have $\mathfrak{M} \in \mathsf{GMJL}_{CS}$ which gives $\phi \in \text{SAT}_{pos}^{\mathsf{GMJL_{CS}}}$.

□

Lemma 3.9. *Let* $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$ *and* $\mathsf{MJL} = \mathsf{GMJL}^{\mathsf{c}}$ *be the corresponding class of classical Mkrtychev models. Let* CS *be a constant specification. Then* $\text{SAT}_{pos}^{\mathsf{GMJL_{CS}}} = \text{SAT}^{\mathsf{MJL_{CS}}} = \text{SAT}_1^{\mathsf{GMJL_{CS}}}$.

Proof. We have $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_{pos} = \mathrm{SAT}^{\mathsf{MJL_{CS}}}$ by Lem. 3.8. By the identification of classical as crisp Gödel-Mkrtychev models, we have $\mathrm{SAT}^{\mathsf{MJL_{CS}}} \subseteq \mathrm{SAT}^{\mathsf{GMJL_{CS}}}_1$ and $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_1 \subseteq \mathrm{SAT}^{\mathsf{GMJL_{CS}}}_{pos}$ follows naturally. □

Using Thm. 2.12 and Thm. 2.10, we now obtain the following theorem summarizing the translated results for many-valued satisfiability.

Theorem 3.10. *Let* $\mathsf{GMJL} \in \{\mathsf{GJ}, \mathsf{GJT}, \mathsf{GJ4}, \mathsf{GLP}, \mathsf{GJ45}, \mathsf{GJT45}\}$ *and let* CS *be a constant specification for* $\mathcal{GJL}_0$*. We have:*

1. *For* $\mathsf{GMJL} \in \{\mathsf{GJ}, \mathsf{GJT}, \mathsf{GJ4}, \mathsf{GLP}\}$ *and* CS *being decidable almost schematic,* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_1$ *and* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_{pos}$ *are in* $\mathsf{co}\text{-}\Pi^p_2 = \Sigma^p_2$.
2. *For* $\mathsf{GMJL} \in \{\mathsf{GJ45}, \mathsf{GJT45}\}$ *and finite* CS*,* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_1$ *and* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_{pos}$ *are decidable.*
3. *For* $\mathsf{GMJL} = \mathsf{GJ4}$ *and decidable schematic* CS *or for* $\mathsf{GMJL} = \mathsf{GLP}$ *and decidable schematically injective axiomatically appropriate* CS*,* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_1$ *and* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_{pos}$ *are* $\mathsf{co}\text{-}\Pi^p_2$*-complete, i.e.* Σ^p_2*-complete.*
4. *For* $\mathsf{GMJL} = \mathsf{GLP}$ *and decidable injective* CS*,* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_1$ *and* $\mathrm{SAT}^{\mathsf{GMJL_{CS}}}_{pos}$ *are* NP*-complete.*

4. Positive validity

Regarding positive validity, we find that although some formulae are classically valid but not 1-valued in the many-valued interpretation (necessarily), they still remain positively-valued. To do this, we induct along the proof in the corresponding proof system, showing that positivity is preserved.

Lemma 4.1. *Let* C *be a class of* GM*-models. Then* $\mathrm{TAUT}^{\mathsf{C}}_{pos}$ *is closed under modus ponens, i.e. if* $\phi, \phi \to \psi \in \mathrm{TAUT}^{\mathsf{C}}_{pos}$*, then* $\psi \in \mathrm{TAUT}^{\mathsf{C}}_{pos}$.

Proof. Let $\phi, \phi \to \psi \in \mathrm{TAUT}^{\mathsf{C}}_{pos}$, i.e. for all $\mathfrak{M} \in \mathsf{C}$: $|\phi|_{\mathfrak{M}}, |\phi \to \psi|_{\mathfrak{M}} > 0$. Thus, for an arbitrary $\mathfrak{M} \in \mathsf{C}$, we have

$$\begin{aligned} |\psi|_{\mathfrak{M}} &\geq |\phi|_{\mathfrak{M}} \odot (|\phi|_{\mathfrak{M}} \Rightarrow |\psi|_{\mathfrak{M}}) \\ &= |\phi|_{\mathfrak{M}} \odot |\phi \to \psi|_{\mathfrak{M}} \\ &> 0. \end{aligned}$$

□

Lemma 4.2. *Let* $\mathsf{GMJL} \in \{\mathsf{GM}, \mathsf{GMT}, \mathsf{GM4}, \mathsf{GMLP}, \mathsf{GM45}, \mathsf{GMT45}\}$ *and* $\mathsf{MJL} = \mathsf{GMJL}^{\mathsf{c}}$ *be the corresponding class of classical Mkrtychev models. Let* CS *be a constant specification for the classical proof system* $\mathcal{JL}$ *associated with* MJL*. Then* $\mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos} = \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$.

Proof. We prove the claim by double inclusion:

⊆: Suppose $\phi \in \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos}$ and let $\mathfrak{M} \in \mathsf{MJL_{CS}}$ be arbitrary. Naturally, we have $\mathfrak{M} \in \mathsf{GMJL^c_{CS}} \subseteq \mathsf{GMJL_{CS}}$ and thus $|\phi|_{\mathfrak{M}} > 0$, i.e. $|\phi|_{\mathfrak{M}} = 1$ and thus by definition we have $\mathfrak{M} \models \phi$. As $\mathfrak{M}$ was arbitrary, we have $\phi \in \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$.

⊇: Let $\phi \in \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$. Thus by completeness, Thm. 2.8, we have $\vdash_{\mathcal{JL}_{CS}} \phi$. We now show by induction on n that

$$\forall n \in \mathbb{N}^* : \forall \alpha \in \mathcal{L}_J : \text{If } \vdash_{\mathcal{JL}_{CS}} \alpha \text{ with a proof of length } n, \text{ then } \alpha \in \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos}.$$

(IB): Let α be arbitrary s.t. $\vdash_{\mathcal{JL}_{CS}} \alpha$ has a proof of length 1. Then α is either an axiom instance of $\mathcal{JL}_{CS}$ or it was obtained by (CS). For the latter, also $\vdash_{\mathcal{GJL}_{CS}} \alpha$ and thus $\alpha \in \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_1 \subseteq \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos}$ by Thm. 2.16. For the former, we either have that α is also an axiom instance of $\mathcal{GJL}_{CS}$, in which case we proceed similarly as in the case for (CS), or $\alpha = \beta \vee \neg\beta$. Let $\mathfrak{M} \in \mathsf{GMJL_{CS}}$ be arbitrary. Then

$$\begin{aligned} |\beta \vee \neg\beta|_{\mathfrak{M}} &= \max\{|\beta|_{\mathfrak{M}}, \sim |\beta|_{\mathfrak{M}}\} \\ &= \begin{cases} 1, & \text{if } |\beta|_{\mathfrak{M}} = 0 \\ |\beta|_{\mathfrak{M}}, & \text{if } |\beta|_{\mathfrak{M}} > 0 \end{cases} \\ &> 0. \end{aligned}$$

Thus, $\alpha \in \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos}$.

(IS): Let $n \in \mathbb{N}^*$, let the claim hold for all $k \leq n$ and let $\alpha \in \mathcal{L}_J$ be arbitrary s.t. $\vdash_{\mathcal{JL}_{CS}} \alpha$ has a proof of length $n+1$, say $(\beta_1, \dots, \beta_n, \alpha)$. Now, either α was obtained as in **(IB)**, in which case we proceed similarly, or α was obtained by (MP) applied to some $\beta_i = \gamma, \beta_j = \gamma \to \alpha$. Now, $(\beta_1, \dots, \beta_i)$, $(\beta_1, \dots, \beta_j)$ are proofs of γ, $\gamma \to \alpha$ with length $\leq n$. By **(IH)**, $\gamma, \gamma \to \alpha \in \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos}$ and by Lem. 4.1, $\alpha \in \mathrm{TAUT}^{\mathsf{GMJL_{CS}}}_{pos}$.

As $\vdash_{\mathcal{JL}_{CS}} \phi$, there is a proof of it of some length and thus by the above claim, $\phi \in \mathrm{TAUT}_{pos}^{\mathsf{GMJL_{CS}}}$.

□

The following theorem is again a translation of the results of Thm. 2.12 and Thm. 2.10. However, crucial for this is the detail of our axiomatization of the classical justification logics. Based on using an extension of $\mathcal{G}$ as an axiomatization of classical logic, we have that every constant specification for some Gödel justification logic is also a constant specification for its classical counterpart. Using this w.r.t. Lem. 4.2, we obtain the following.

Theorem 4.3. *Let* $\mathsf{GMJL} \in \{\mathsf{GJ}, \mathsf{GJT}, \mathsf{GJ4}, \mathsf{GLP}, \mathsf{GJ45}, \mathsf{GJT45}\}$ *and let* CS *be a constant specification for* $\mathcal{GJL}_0$. *We have:*

(1) For $\mathsf{GMJL} \in \{\mathsf{GJ}, \mathsf{GJT}, \mathsf{GJ4}, \mathsf{GLP}\}$ *and* CS *being decidable almost schematic,* $\mathrm{TAUT}_{pos}^{\mathsf{GMJL_{CS}}}$ *is in* Π_2^p.
(2) For $\mathsf{GMJL} \in \{\mathsf{GJ45}, \mathsf{GJT45}\}$ *and finite* CS, $\mathrm{TAUT}_{pos}^{\mathsf{GMJL_{CS}}}$ *is decidable.*
(3) For $\mathsf{GMJL} = \mathsf{GJ4}$ *and decidable schematic* CS *or for* $\mathsf{GMJL} = \mathsf{GLP}$ *and decidable schematically injective axiomatically appropriate* CS, $\mathrm{TAUT}_{pos}^{\mathsf{GMJL_{CS}}}$ *is* Π_2^p*-complete.*
(4) For $\mathsf{GMJL} = \mathsf{GLP}$ *and decidable injective* CS, $\mathrm{TAUT}_{pos}^{\mathsf{GMJL_{CS}}}$ *is* co-NP*-complete.*

5. 1-validity

The case of 1-validity over model classes remains intricate as satisfiability and validity are not dually related as in the classical case. It remains of vital importance however, as we have $TAUT_1^{\mathsf{GMJL_{CS}}} = Th_{\mathcal{GJL}_{CS}}$. At first, we get a similar trivial lower bound as in the case of Thm. 2.11.

Lemma 5.1 (P. [24])**.** *Let* $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$ *and let* CS *be a constant specification for* $\mathcal{GJL}_0$. *Then for any* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_0$: $\Gamma \vdash_{\mathcal{G}} \phi$ *if, and only if* $\Gamma \vdash_{\mathcal{GJL}_{CS}} \phi$.

Lemma 5.2 (Hájek [12])**.** $Th_{\mathcal{G}} = \{\phi \in \mathcal{L}_0 \mid \vdash_{\mathcal{G}} \phi\}$ *is* co-NP*-complete.*

Using this lemma, we then obtain the following formulation.

Theorem 5.3. *Let* $\mathcal{GJL}_0 \in \{\mathcal{GJ}_0, \mathcal{GJT}_0, \mathcal{GJ}4_0, \mathcal{GLP}_0, \mathcal{GJ}45_0, \mathcal{GJT}45_0\}$ *and* CS *be a constant specification for* $\mathcal{GJL}_0$. *Then* $Th_{\mathcal{GJL}_{CS}}$ *is* co-NP*-hard.*

We can however extend these lower bounds further. For this, we employ a crisp projection using double negation similar as in Hájek's [12].

Definition 5.4. We define the map $\cdot^{\neg\neg} : \mathcal{L}_J \to \mathcal{L}_J$ as follows:

- $\bot \mapsto \bot$, $\top \mapsto \top$,
- $p \mapsto \neg\neg p$ for $p \in Var$,
- $(\phi \land \psi) \mapsto (\phi^{\neg\neg} \land \psi^{\neg\neg})$,
- $(\phi \to \psi) \mapsto (\phi^{\neg\neg} \to \psi^{\neg\neg})$,
- $t : \phi \mapsto \neg\neg t : \phi$.

Lemma 5.5. *If* $\mathfrak{M} \in \mathsf{GM^c}$, *then for all* $\phi \in \mathcal{L}_J$: $|\phi^{\neg\neg}|_{\mathfrak{M}} = |\phi|_{\mathfrak{M}}$.

Proof. First, if $\mathfrak{M} = \langle \mathcal{E}, e \rangle \in \mathsf{GM^c}$, then for any $\phi \in \mathcal{L}_J$, $|\phi|_{\mathfrak{M}} \in \{0, 1\}$. The proof proceeds by induction on ϕ, where we just note that $\sim\sim 0 = 0$ and $\sim\sim 1 = 1$. □

Lemma 5.6. *For any* $\mathfrak{M} \in \mathsf{GM}$ *and any* $\phi \in \mathcal{L}_J$: $|\phi^{\neg\neg}|_{\mathfrak{M}} \in \{0, 1\}$.

Proof. The proof proceeds by induction on ϕ, noting that $\odot, \Rightarrow$ are $\{0,1\}$-valued when restricted to arguments from $\{0,1\}^2$ and that for $x \in [0,1]$, we have $\sim\sim x \in \{0,1\}$. □

Lemma 5.7. *For any* $\phi \in \mathcal{L}_J$: $\phi \in \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$ *iff* $\phi^{\neg\neg} \in \mathrm{TAUT}_1^{\mathsf{GMJL_{CS}}}$.

Proof. $\Leftarrow$**:** Let $\phi^{\neg\neg} \in \mathrm{TAUT}_1^{\mathsf{GMJL_{CS}}}$. Then $\phi^{\neg\neg} \in \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$ (either directly or by Lem. 4.2) and thus by Lem. 5.5, we have $\phi \in \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$ as $\mathsf{MJL_{CS}} = \mathsf{GMJL^c_{CS}}$.

$\Rightarrow$**:** Let $\phi^{\neg\neg} \notin \mathrm{TAUT}_1^{\mathsf{GMJL_{CS}}}$, i.e. $\exists \mathfrak{M} = \langle \mathcal{E}, e \rangle \in \mathsf{GMJL_{CS}} : |\phi^{\neg\neg}]_{\mathfrak{M}} < 1$. By Lem. 5.6, we thus have $|\phi^{\neg\neg}]_{\mathfrak{M}} = 0$. We now define $\widetilde{\mathfrak{M}} = \langle \widetilde{\mathcal{E}}, \widetilde{e} \rangle$ by

- $\widetilde{\mathcal{E}}(t, \phi) = \sim\sim \mathcal{E}(t, \phi)$,
- $\widetilde{e}(p) = \sim\sim e(p)$.

Then $\widetilde{\mathfrak{M}} \in \mathsf{GMJL^c_{CS}} = \mathsf{MJL_{CS}}$ and for any $\phi \in \mathcal{L}_J$, we have $|\phi^{\neg\neg}|_{\mathfrak{M}} = |\phi|_{\widetilde{\mathfrak{M}}}$ and thus we have $|\phi|_{\widetilde{\mathfrak{M}}} = |\phi^{\neg\neg}|_{\mathfrak{M}} = 0$, i.e. $\widetilde{\mathfrak{M}} \not\models \phi$ and thus $\phi \notin \mathrm{TAUT}^{\mathsf{MJL_{CS}}}$.

□

Lemma 5.8. $\mathrm{TAUT}^{\mathsf{MJL_{CS}}}$ *can be reduced to* $\mathrm{TAUT}_1^{\mathsf{GMJL_{CS}}}$ *in linear time over the length of the formula.*

Proof. We may compute $\cdot^{\neg\neg}$ by an algorithm that moves iteratively from left to right over the symbols of the formula, replacing every p by a $\neg\neg p$ and every $t : \phi$ by a $\neg\neg t : \phi$ and leaving every other symbol put. □

Thus, we have that the complexity of the theorem sets of the Gödel justification logics is always bounded below by the complexity of the corresponding classical justification logic. This can be reformulated into the following theorem, utilizing Thm. 4.3.

Theorem 5.9. *For* $\mathsf{GMJL} = \mathsf{GJ4}$ *(*$\mathcal{GJL}_0 = \mathcal{GJ}4_0$*) and decidable schematic* CS *or for* $\mathsf{GMJL} = \mathsf{GLP}$ *(*$\mathcal{GJL}_0 = \mathcal{GLP}_0$*) and decidable schematically injective axiomatically appropriate* CS*,* $\mathrm{TAUT}_1^{\mathsf{GMJL}_{\mathsf{CS}}} = Th_{\mathcal{GJL}_{CS}}$ *is* Π_2^p*-hard.*

Note, that part (2) from Thm. 2.12 does not give anything new, as co-NP-hardness is already obtained by the trivial lower bound. In general, as exact or lower bounds on classical justification logics are sparse, the above theorem is sparse as well. However, using Lem. 5.8, future progress into lower bounds on decidability of justification logics also updates Thm. 5.9.

In terms of upper bounds, we think that methods from [16, 17] can be transferred appropriately to the many-valued cases and would thereby result in a similar statement as in Thm. 4.3 for the case of 1-validity. This is however left for future work. Further, it shall also be interesting as to whether the methods of Studer's work [26] can be adapted to show decidability results for the theorems of $\mathcal{GJ}45_{CS}$ and $\mathcal{GJT}45_{CS}$.

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